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MATRICES : SOME NEW PROPERTIES. / SB'S THEOREMS (SPECTRUM)

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ABSTRACT

Matrix---not only the arrays of numbers but also has been used as a tool for many calculations in various subjects. Its inverse, eigen values, eigen vectors are of great importance to know its characters. In this paper I have discussed about some new properties of Hermitian, Skew Hermitian matrices, diagonalisation, eigen values, eigen vectors, spectrum, which will open up a new horizon to the students of Mathematics. Also, in this paper I have authored two totally new theorems for students and researchers . SB's Theorem 1 is on Normality of a block diagonal matrix and SB's Theorem 2 is on Spectrum of eigen values. These ideas came to me in course of teaching. Hope, these two theorems will be of great help for the students of Physics and Chemistry as well.

1. INTRODUCTION

Recall

1) A complex square matrix A will be Hermitian if it's (rs)th element and (sr)th element be complex conjugate of one another i.e. $A = [\bar{A}]^{T} = A^{H}$.
2) The entries on the main diagonal of any Hermitian matrix are real .
3) A complex square matrix A is Unitary if $AA^{H} = I$.
4) A complex square matrix A is normal iff $A^{H}A = AA^{H}$.
5) A matrix $A \in \mathbb{C}^{n\times n}$ is normal if and only if it is unitarily diagonalizable i.e.

$A = UDU^{H}$, the columns of unitary matrix U are unit eigen vectors and the diagonal elements of diagonal matrix D are the eigen values of A that correspond to those eigen vectors .

6) Among complex matrices ,all unitary , Hermitian and skew Hermitian matrices are normal.

However, it is not the case that all normal matrices are either unitary or (skew)—Hermitian .

Being a student of Mathematics I have tried to share my own thoughts and ideas with other students through this paper .

Property 1- If A and B are Hermitian matrices then aA+ bB is also Hermitian for all real scalars a and b .
Proof: Since A and B are Hermitian matrices we have ,

$A^H = A \; and \; B^H = B. \ldots \ldots \ldots \ldots \ldots .. (i)$

Now, $(aA + bB)^H = \bar{a}A^H + \bar{b}B^H$

i.e $(aA + bB)^H = aA^H + bB^H$,

Since a ,b are real scalars .

i.e . $(aA + bB)^H = aA + bB \quad by \quad (i)$

It proves that $(aA + bB) \; is \; Hermitian$.

Property 2- A is a Hermitian matrix if and only if iA(or-iA) is skew Hermitian matrix .
Proof: Let A be a Hermitian matrix, then $A^H = A$

So $(iA)^H = \bar{\imath}A^H = -iA^H = -(iA)$

Which proves that iA is skew Hermitian .
Conversely, let $(iA)^H = -iA$,
or , $\bar{\imath}A^H = -iA$
Or , $-A^H = -iA$
or , $A^H = A$.

So , A is Hermitian.
Hence proved .

Property 3- The inverse of an invertible skew Hermitian matrix is skew Hermitian.
Proof : Let A be an invertible skew Hermitian matrix i.e. $A^H = -A$ and if *B* be the inverse of *A* then *A.B* = *I* (*Identity matrix*) *i.e .* $B = A^{-1}$ *. Now* $B^H = (A^{-1})^H = (A^H)^{-1}$

$= (-A)^{-1} = -A^{-1} = -B$.

Hence it is proved that B is skew Hermitian matrix .

Property 4- *Product of two skew Hermitian matrices A and B is Hermitian if and only if AB=BA*
Proof: Here A and b be two skew Hermitian matrices *i.e.* $A^H = -A \; and \; B^H = -B$.
Now, $(AB)^H = \overline{(AB)^T} = \overline{(B^T A^T)}$
$(AB)^H = B^H A^H = BA$.
Thus ,$(AB)^H = AB \; if \; and \; only \; if \; AB = BA$.
Hence proved.

Property 5- *If A and B are normal matrices with AB=BA then A-B is also normal*
Proof: *Here we have* $AA^H = A^H A, B^H B = BB^H$ *since A and B are normal* .

Now, $(A-B)(A-B)^H$
$= (A-B)(A^H-B^H)$
$= AA^H - AB^H - BA^H + BB^H$
$= A^HA - A^HB - B^HA + B^HB$

$$\because \left\{ \begin{array}{l} AA^H = A^HA, BB^H = B^HB \quad (\text{ A and B are normal}) \\ (BA^H = A^HB, AB^H = B^HA \text{ by Fuglede'Theorem}) \end{array} \right\}$$

$= (A^H - B^H)(A-B)$
$= (A-B)^H(A-B)$, So $A-B$ is normal .

Hence proved .

Note: **Fuglede's Theorem :** *If bounded operator T and S on a Hillbert space commute and S is normal then T and* S^H *commute* .

Property 6- If A be a square matrix of order n then the sum of products of eigen values taken r at a time (r<n) is equal to the [sum of principal minors of A of order r] .

Proof: Let A be the square matrix [aij] of order n. If $\alpha_1, \alpha_2, \dots \dots \alpha_n$ be the eigen values of A, then –

$$|A-\lambda I| = (-1)^n(\lambda-\alpha_1)(\lambda-\alpha_2) \dots \dots \dots \dots \dots \dots . . (\lambda-\alpha_n) \text{ ....(ii)}$$

The coefficient of λ^{n-r} in | A-λI | in left hand side is $(-1)^n$ [sum of the principal minors of A of order r] and the coefficient of λ^{n-r} on the right hand side of (ii) is $(-1)^{n+r}$ [the sum of the products of eigen values taken r at a time](r < n)

Therefore, the sum of the products of the eigen values taken r at a time = sum of the principal minors of A of order r.

Hence proved .

Lemma (1): If A ∈ $\mathbb{C}^{n\times n}$ be a block diagonal matrix ,

$$A = \begin{pmatrix} A_{11} & 0 \text{ ....} & 0 \\ 0 & A_{22} \text{ ....} & 0 \\ 0 & 0 \text{ ....} & 0 \\ 0 & \dots\dots\dots\dots\dots\dots\dots\dots . & 0 \\ .. & .. \; .. \; \dots\dots . \; .. \; .. & \dots\dots \\ 0 & .. \; .. \; \dots\dots\dots\dots\dots . & A_{mm} \end{pmatrix}$$

Then A is diagonalizable if and only if every A_{ii} is diagonalizable for 1 ≤ i ≤ m .

Proof : The argument is straight forward and is omitted .

SB' s Theorem (1) : Statement :

Let $A \in \mathbb{C}^{n\times n}$ be a block diagonal matrix

$$A = \begin{pmatrix} A_{11} & 0 \ldots\ldots\ldots\ldots & 0 \\ 0 & A_{22} \ldots\ldots\ldots\ldots & 0 \\ & 0 \ldots\ldots\ldots\ldots & 0 \\ 0 & \ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots & \ldots\ldots\ldots & \ldots \\ \ldots & \ldots\ldots\ldots & \ldots \\ 0 & \ldots\ldots\ldots\ldots\ldots\ldots & A_{mm} \end{pmatrix}$$

Then A is normal if and only if every matrix A_{ii} is normalfor $1 \le i \le m$.

Proof: Let every matrix A_{ii} is normal which automatically implies that each A_{ii} is unitarily diagonalizable. Now , by the lemma 1 , the matrix A is unitarily diagonalizable which in turn implies that the matrix A is normal.

The proof of the converse implication is very simple and so left for the reader. (proved)

Lemma (2): Let A be a Hermitian matrix and $\lambda_1 \ge \lambda_2 \ge$ ------$\ge \lambda_n$ be its eigen values with the orthonormal eigen vectors u_1, u_2 ,------- u_n respectively . Let us define a subspace

$S = (u_p$,------$u_q)$ where $1 \le p \le q \le n$ .If $x \in S$ and $\|x\|_2 = 1$, we have $\lambda_q \le x^H A x \le \lambda_p$

Proof : *Let x be a unit vector in S , then $x = v_p u_p +$ ----- $+ v_q u_q$ so that $x^H u_i = \overline{v_i}$. for $p \le i \le q$ .Since $\|x\|_2 = 1$, we have $|v_p|^2 +$ ------$+ |v_q|^2 = 1$. This allows us to write -----------------*

$x^H Ax = x^H(v_p Au_p +$ -------------- $+ v_q Au_q)$

$= x^H(v_p \lambda_p u_p +$ ------------ $+ v_q \lambda_q u_q)$

$= x^H(|v_p|^2 \lambda_p +$ ---------- $+ |v_p|^2 \lambda_q)$.

Since $|v_p|^2 +$ ------$+ |v_q|^2 = 1$, the desired inequality follows .

SB's Theorem (2) (spectrum) :

Let $A, B \in \mathbb{C}^{n\times n}$ be two Hermitian matrices . Define , E =A+B . Suppose that the eigen values of A , B , E are $\alpha_1 \ge \alpha_2 \ge$ ------- $\ge \alpha_n$; $\beta_1 \ge \beta_2 \ge - - - \ge \beta_n$ and $\mathcal{E}_1 \ge \mathcal{E}_2 \ge - - - - \ge \mathcal{E}_n$ *respectively .*

Then we have $\boldsymbol{\mathcal{E}_n \le \beta_i + \alpha_i \le \mathcal{E}_1}$

Proof : Since A and B are two Hermitian matrices , E is also Hermitian. So all matrices involved have real eigen values .

By Courant –Fisher Theorem--------------

$$\mathcal{E}_k = \min_{w} \max_{x} \{x^H E x \mid \|x\|_2 = 1 \text{ and } w_i^H x = 0 \text{ for } 1 \le i \le k-1\} \text{ where } w = \{w_1, w_2, \text{-----} w_{k-1}\}$$

$$\text{So } \mathcal{E}_k \le \max_{x} x^H E x = \max_{x} (x^H A x + x^H B x) \text{ ----------------------(iii)}$$

Let us take unitary matrix U such that $U^H AU = diag(\alpha_1, \alpha_2, \text{--------------} \alpha_n)$.
Let $w_i = Ue_i$ *, for* $1 \le i \le k\text{-}1$ *. Then we have ------------*

$$w_i^H x = e_i^H U^H x = 0 \quad \text{for } 1 \le i \le k\text{-}1 .$$

Let us define $Y = U^H x$ *and since U is an unitary matrix ,* $\|Y\|_2 = 1$ *and* $\|x\|_2 = 1$ *,*
We have $e_i^H Y = y_i = 0$ *for* $1 \le i \le k$ *.*
Therefore $\sum_{i=k}^{n} y_i^2 = 1$ *. This in turn implies that*

$$x^H A x = Y^H U^H A U Y = \sum_{i=k}^{n} \alpha_i y_i^2 \le \alpha_k .$$

Following the same process we can show that $x^H B x \le \beta_k$

So , from (iii) we have $\mathcal{E}_k \le \alpha_k + \beta_k$ *----------------------(iv)*

By the lemma (2) and from inequality –(iv) we can conclude the desired inequality

$$\mathcal{E}_n \le \beta_i + \alpha_i \le \mathcal{E}_1 .$$

Hence proved .

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