ON AN EXPANSION FORMULA FOR THE MULTIVARIABLE $I$-FUNCTION INVOLVING GENERALIZED LEGENDRE’S ASSOCIATED FUNCTION

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ABSTRACT

The authors have established a new expansion formula for multivariable $I$-function due to Prasad [5] in terms of products of the multivariable $I$-function and the generalized Legendre’s associated function due to Meulenbeld [3]. Some special cases are given in the last.

Keywords:
Multivariable $I$-function, Generalized Legendre’s associated function, Multivariable $H$-function.

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1. INTRODUCTION

The multivariable $I$-function introduced by Prasad [5] will be define and represent it in the following manner:

$$I[z_1,...,z_r] = I^{0,n_1,...,0,n_r}(m^{r'),r^{r'})\prod p^{r},q^{r'}=...\left(\frac{d^{r'),r^{r'}}{d^{r'),r^{r'}}}\right)$$

$$\left[z_{1},...,z_{r}\right]^{(a_{1},a_{2},...,a_{r};b_{1},b_{2},...,b_{r};c_{1},c_{2},...,c_{r};d_{1},d_{2},...,d_{r};e_{1},e_{2},...,e_{r};f_{1},f_{2},...,f_{r})}$$

$$= \frac{1}{(2\pi i)^{r}}\lim_{L \to \infty} \left[\int_{L_{1}}^{L} \cdots \int_{L_{r}}^{L} \phi_{1}(s_{1})\cdots\phi_{r}(s_{r})\psi(s_{1},...,s_{r})z_{1}^{s_{1}}\cdots z_{r}^{s_{r}}ds_{1}\cdots ds_{r}\right] (1.1)$$
Where

\[ w = \sqrt{(-1)} \]

\[
\phi(s_i) = \frac{\prod_{j=1}^{p} \Gamma(b_j^{(i)} - \beta_j^{(i)} s_i) \prod_{j=1}^{q} \Gamma(1 - b_j^{(i)} s_i) \prod_{j=1}^{w} \Gamma(1 - a_j^{(i)} s_i)}{\prod_{j=m+1}^{r} \Gamma(1 - b_j^{(i)} s_i) \prod_{j=1}^{w} \Gamma(a_j^{(i)} s_i)} \quad \forall i \in (1, 2, \ldots, r) \quad (1.2)
\]

\[
\varphi(s_1, \ldots, s_r) = \prod_{j=1}^{n_1} \Gamma\left(1 - a_{2j}^{(i)} + \sum_{i=1}^{2} \alpha_{2j}^{(i)} s_i\right) \prod_{j=1}^{n_2} \Gamma\left(1 - a_{3j}^{(i)} + \sum_{i=1}^{3} \alpha_{3j}^{(i)} s_i\right) \ldots \prod_{j=1}^{n_r} \Gamma\left(1 - a_{nj}^{(i)} + \sum_{i=1}^{r} \alpha_{nj}^{(i)} s_i\right)
\]

\[
\prod_{j=n_{r+1}}^{p} \Gamma\left(a_{nj} - \sum_{i=1}^{r} \alpha_{nj}^{(i)} s_i\right) \prod_{j=n_{r+1}}^{q_1} \Gamma\left(1 - b_{nj} - \sum_{i=1}^{r} \beta_{nj}^{(i)} s_i\right) \prod_{j=n_{r+1}}^{q_r} \Gamma\left(1 - b_{nj} - \sum_{i=1}^{r} \beta_{nj}^{(i)} s_i\right)
\]

\[
\frac{1}{\prod_{j=1}^{n_{r+1}} \Gamma\left(a_{nj} - \sum_{i=1}^{r} \alpha_{nj}^{(i)} s_i\right)} \prod_{j=1}^{n_{r+1}} \Gamma\left(1 - a_{nj} + \sum_{i=1}^{r} \alpha_{nj}^{(i)} s_i\right)
\]

\[
\prod_{j=n_{r+1}}^{p} \Gamma\left(a_{nj} - \sum_{i=1}^{r} \alpha_{nj}^{(i)} s_i\right) \prod_{j=n_{r+1}}^{q_1} \Gamma\left(1 - b_{nj} - \sum_{i=1}^{r} \beta_{nj}^{(i)} s_i\right) \prod_{j=n_{r+1}}^{q_r} \Gamma\left(1 - b_{nj} - \sum_{i=1}^{r} \beta_{nj}^{(i)} s_i\right)
\]

\[
\alpha_j^{(i)}, \beta_j^{(i)}, \alpha_{kj}^{(i)}, \beta_{kj}^{(i)} (i = 1, \ldots, r) (k = 1, \ldots, r) \text{ are positive numbers,}
\]

\[
a_j^{(i)}, b_j^{(i)} (i = 1, \ldots, r), a_{kj}, b_{kj} (k = 2, \ldots, r) \text{ are complex numbers and here}
\]

\[
m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)} (i = 1, \ldots, r), n_k, p_k, q_k (k = 2, \ldots, r) \text{ are non-negative integers where}
\]

\[
0 \leq m^{(i)} \leq q^{(i)}, 0 \leq n^{(i)} \leq p^{(i)}, q^k \geq 0, 0 \leq n_k \leq p_k. \text{ Here } (i) \text{ denotes the numbers of dashes. The}
\]

contours \( L_i \) in the complex \( s_i \)-plane is of the Mellin-Barnes type which runs from \(-w\infty\) to \(+w\infty\) with indentations, if necessary, to ensure that all the poles of \( \Gamma\left(b_j^{(i)} - \beta_j^{(i)} s_i\right) \) \((j = 1, \ldots, m^{(i)})\) are

separated from those of \( \Gamma\left(1 - a_{nj} + \sum_{i=1}^{r} \alpha_{nj}^{(i)} s_i\right) \) \((j = 1, \ldots, n_r)\).

For further details and asymptotic expansion of the \( I \)-function one can refer by Prasad [5].

In what follows, the multivariable \( I \)-function defined by Prasad [5] will be represented in the contracted notation:

\[
I_{0, n_1; \ldots; 0, n_r; \{m^{(r)}, n^{(r)}\}; \{m^{(r)}, n^{(r)}\}} \left[z_1, \ldots, z_r\right]
\]

Or simply by \( I[z_1, \ldots, z_r]\).

According to the asymptotic expansion of the gamma function, the counter integral (1.1) is absolutely convergent provided that
\[ \arg z_i < \frac{1}{2} \pi U, U_i > 0 \quad ; \quad i = 1, 2, \ldots, r \] (1.4)

Where

\[
U_i = \sum_{j=1}^{d} \alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{m} \alpha_j^{(i)} + \sum_{j=1}^{m} \beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{r} \beta_j^{(i)}
\]

\[+ (\sum_{j=1}^{n_i} \alpha_j^{(i)} - \sum_{j=n_i+1}^{n} \alpha_j^{(i)}) + (\sum_{j=1}^{n_i} \alpha_j^{(i)} - \sum_{j=n_i+1}^{n} \alpha_j^{(i)})
\]

\[+ \ldots + (\sum_{j=1}^{n_r} \alpha_j^{(i)} - \sum_{j=n_r+1}^{n} \alpha_j^{(i)})
\]

\[- (\sum_{j=1}^{q_1} \beta_j^{(i)} + \sum_{j=1}^{q_2} \beta_j^{(i)} + \ldots + \sum_{j=1}^{q_r} \beta_j^{(i)}) \] (1.5)

The asymptotic expansion of the \( I \)-function has been discussed by Prasad [5]. His results run as follow:

\[ I[z_1, \ldots, z_r] = 0(\max\{|z_1|, \ldots, |z_r|\} \to 0 \]

Where

\[ \alpha_i = \min \Re(b_j^{(i)}/\beta_j^{(i)}) \quad ; \quad j = 1, \ldots, m^{(i)} \quad ; \quad i = 1, \ldots, r \] (1.6)

And

\[ I[z_1, \ldots, z_r] = 0(\max\{|z_1|, \ldots, |z_r|\} \to \infty \]

Where

\[ \beta_i = \max \Re\left(\frac{d_j^{(i)}-1}{\alpha_j^{(i)}}\right) \quad ; \quad j = 1, \ldots, n^{(i)} \quad ; \quad i = 1, \ldots, r \] (1.7)

The details of the function can be found in the paper of Prasad [5].

In this paper we will evaluate an integral involving generalized associated Legendre’s function and the multivariable \( I \)-function due to Prasad [5] and apply it in deriving an expansion for the multivariable \( I \)-function in series of products of associated Legendre’s function and the multivariable \( I \)-function.

2. THE INTEGRAL

The integral to be evaluated is:
The integral (2.1) is valid under the following set of conditions:

(i) \( (\alpha_i, \beta_i) > 0; \forall i \in \{1, 2, \ldots, r\}; k - \frac{u - v}{2} \) is a positive integer, \( k \) is an integer \( \geq 0 \).

(ii) \( \text{Re} \left( \rho - u + \sum_{i=1}^{r} \alpha_i \frac{b^{(i)}_j}{\beta^{(i)}_j} \right) > -1; \text{Re} \left( \sigma + v + \sum_{i=1}^{r} \beta_i \frac{b^{(i)}_j}{\beta^{(i)}_j} \right) > -1; (j = 1, 2, \ldots, m_i; i = 1, 2, \ldots, r) \)

And the conditions given in (1.4) to (1.7) are also satisfied.

**Proof:** On expressing the multivariable \( I \)-function in the integrand as a multiple Mellin-Barnes type integral (1.1) and inverting the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, the value of the integral

\[
\frac{1}{k - \frac{u - v}{2}} \int_{-1}^{1} (1 - x)^{\rho - u} (1 + x)^{\sigma + v} P^{\mu, \nu}_{k - \frac{u - v}{2}}(x) dx
\]

\[
= 2^{\rho - u + v + 1} \sum_{l=0}^{\infty} \frac{(-1)^l (v - u + k + 1)}{l!} \Gamma(l - u + t) (t!) \int_{0}^{\infty} \cdots \int_{0}^{\infty} \prod_{i=1}^{r} \left( \prod_{j=1}^{m_i} \left( \frac{m^{(i)}_{j}}{n^{(i)}_{j}} \right) \right)
\]

\[
\left[ \left( a_{j1}, a_{j2}, \ldots, b_{j1}, b_{j2}, \ldots \right) \right] \left( a_{j1}, a_{j2}, \ldots, b_{j1}, b_{j2}, \ldots \right) \left( \cdots \right) \left( \cdots \right)
\]

\[
(2\pi)^r \int_{L_{i_1}} \cdots \int_{L_{s}} \omega(s_1, \ldots, s_r) \sum_{i=1}^{\infty} \left\{ \phi_i(s_i) z^{\xi}_{i} \right\}
\]

\[
\prod_{i=1}^{r} \int_{-1}^{1} (1 - x)^{\rho - u} (1 + x)^{\sigma + v} \frac{\beta_i}{\beta^{(i)}_j} \frac{\alpha_i}{\alpha^{(i)}_j} \frac{1}{\xi_i} dx
\]

\[
P^{\mu, \nu}_{k - \frac{u - v}{2}}(x) dx
\]

On evaluating the \( x \)-integral with the help of the integral ([4], p.343, eq. (38)):
\[ 2^{p^+q^m-n^2} \frac{\Gamma\left(\rho - \frac{m}{2} + 1\right) \Gamma\left(\sigma + n + 1\right)}{\Gamma(1-m) \Gamma\left(\rho + \sigma - \frac{m-n}{2} + 2\right)} \times \, _3F_2\left(-k, n-m+k+1, \rho - \frac{m}{2} + 1; 1-m, \rho - \sigma - \frac{m-n}{2} + 2; 1\right) \] (2.2)

Provided that \( \Re\left(\rho - \frac{m}{2}\right) > -1; \Re\left(\sigma + \frac{n}{2}\right) > -1 \) and interpreting the result with the help of (1.1), the integral (2.1) is established.

### 3. EXPANSION THEOREM

Let the following conditions be established:

(i) \( \beta_1, \ldots, \beta_r > 0; \alpha_1, \ldots, \alpha_r \geq 0 (\text{or } \beta_1, \ldots, \beta_r \geq 0; \alpha_1, \ldots, \alpha_r > 0) \);

(ii) \( m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)} (i = 1, \ldots, r), n_k, p_k, q_k (k = 2, \ldots, r) \) are non-negative integers where \( 0 \leq m^{(i)} \leq q^{(i)}, 0 \leq n^{(i)} \leq p^{(i)}, q^{(i)} \geq 0, 0 \leq n_k \leq p_k \) and the conditions given by (1.4) to (1.7) are also satisfied.

(iii) \( \Re(u) > -1, \Re(v) > -1, \Re\left(\rho - u + \sum_{i=1}^{r} \alpha_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > -1; \Re\left(\sigma + v + \sum_{i=1}^{r} \beta_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > -1; (j = 1, 2, \ldots, m_i; i = 1, 2, \ldots, r) \).

Then the following expansion formula holds:

\[
(1-x)^{\rho-u/2} (1+x)^{\sigma+v/2} I \left[ (1-x)^{\alpha_i} (1+x)^{\beta_i} z_i, \ldots, (1-x)^{\alpha_r} (1+x)^{\beta_r} z_r \right]
\]

\[
= 2^{\rho+\sigma} \sum_{N=0}^{\infty} \sum_{\mu=0}^{N} \frac{(2N-u+v+1)\Gamma(N-u+1)\Gamma(1+v-u+N+\mu)(-N)_{\mu}}{N! \mu! \Gamma(1+v+N)\Gamma(1-u+\mu)} \left( \frac{p^{(i)}}{q^{(i)}} \right)^{m^{(i)}-n^{(i)}} \prod_{j=1}^{r} \frac{\beta_j^{(i)}}{\beta_j^{(i)}}
\]

\[
\left[ \begin{array}{c}
m^{(i)} \beta_j^{(i)} z_j \\
q^{(i)} \beta_j^{(i)} z_j \\
\vdots \\
p^{(i)} \beta_j^{(i)} z_j \\
\end{array} \right] = \left( \begin{array}{c}
\alpha_j \\
\beta_j \\
\vdots \\
\gamma_j \\
\end{array} \right) \cdot \left( \begin{array}{c}
\alpha_j \\
\beta_j \\
\vdots \\
\gamma_j \\
\end{array} \right)
\]
Provided that $\Re(u), 1, \Re(v) < 1$; we obtain

$$
C_k = \frac{2^{\nu + \sigma} (2k - u + v + 1) \Gamma (k - u + 1)}{k! \Gamma (k + v + 1)} \sum_{\mu=0}^{k} \frac{(-k)_\mu \Gamma (k - u + v + \mu + 1)}{\mu! \Gamma (k - u + \mu)}
$$

Now on substituting the values of $C_k$ in (3.2), the result follows.

4. SPECIAL CASES

If in (2.1), we put $n_2 = ... = n_{r-1} = 0 = p_2 = ... = p_{r-1}, q_2 = ... = q_{r-1} = 0$, the multivariable $I$ - function reduces to multivariable $H$ -function and we get result given by Saxena and Ramawat [6]
Provided all the conditions given with (3.1) and the conditions ([7], p.252-253, eq. (c.4), (c.5) and (c.6)) are satisfied.

For \( n = 0 = p, q = 0 \), the multivariable \( H \)-function breaks up into a product of \( r \) \( H \)-function and consequently, (4.1) reduces to

\[
(1-x)^{\rho_u - \frac{u}{2}} (1+x)^{\sigma_v} H \left[ (1-x)^{\alpha} (1+x)^{\beta} z_1, \ldots, (1-x)^{\alpha} (1+x)^{\beta} z_r \right]
\]

\[
= 2^{\rho_u + \sigma_v} \sum_{N=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{(2N - u + v + 1) \Gamma(N - u + 1) \Gamma(1 + v - u + N + \mu) (-N)_{\mu}}{N! \mu! \Gamma(1 + v + N) \Gamma(1 - u + \mu)}
\]

\[
P_{N - \frac{u + v}{2}}^{\mu, \nu}(x) H_{N - \frac{u + v}{2}}^{\mu, \nu}(x)
\]

\[
= 2^{\rho_u + \sigma_v} \sum_{N=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{(2N - u + v + 1) \Gamma(N - u + 1) \Gamma(1 + v - u + N + \mu) (-N)_{\mu}}{N! \mu! \Gamma(1 + v + N) \Gamma(1 - u + \mu)}
\]

\[
P_{N - \frac{u + v}{2}}^{\mu, \nu}(x) H_{N - \frac{u + v}{2}}^{\mu, \nu}(x)
\]

For \( r = 1 \), (4.2) gives rise to the result due to Anandani [1].

**5. REFERENCES**


