APPLICATIONS OF EDGE COLORING OF GRAPHS WITH RAINBOW NUMBERS PHENOMENA

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ABSTRACT

This paper studies the Rainbow Ramsey Number for a non-empty graph and the main results are 1. The Rainbow Ramsey Number of a graph F with out isolated vertices is defined if and only if F is a forest. 2. The Rainbow Ramsey Number of two graphs F₁ and F₂ with out isolated vertices is defined if and only if F₁ is a star or F₂ is a forest.

Mathematics Subject Classification 2000: 05CXX, 05C55, 05DXX, 05D10, 04XX, 04A10

Keywords:
Rainbow Ramsey Number, forest, isolated vertices, star.


1. INTRODUCTION

Basically in an edge-colored graph G that if there is a sub graph F of G all of whose edges are colored the same, then F is referred to as a monochromatic F. On the other hand, if all edges of F are colored differently, then F is referred to as a rainbow F.

2. DEFINITION

For a nonempty graph F, the Rainbow Ramsey Number RR (F) of F as the smallest positive integer n such that if each edge of the complete graph Kₙ is colored from any set of colors, then either a monochromatic F or a rainbow F is produced.
Let \( \{v_1, v_2, \ldots, v_n\} \) be the vertex set of a complete graph \( K_n \). An edge coloring of \( K_n \) using positive integers for colors is called a **minimum coloring** if two edges \( v_i v_j \) and \( v_k v_l \) are colored the same if and only if
\[
\min \{i, j\} = \min \{k, l\}
\]
while an edge coloring of \( K_n \) is called a **maximum coloring** if two edges \( u_i u_j \) and \( u_k u_l \) are colored the same if and only if
\[
\max \{i, j\} = \max \{k, l\}
\]

2.1. **Definition:** A graph with out cycles is a forest.

2.2. **Theorem:** Let \( F \) be a graph without isolated vertices. The **Rainbow Ramsey number** \( RR(F) \) is defined if and only if \( F \) is a forest.

Let \( F \) be a graph of order \( p \geq 2 \). First we show that \( RR(F) \) is defined only if \( F \) is a forest. Suppose that \( F \) is not a forest. Thus \( F \) contains a cycle \( C \), of length \( k \geq 3 \) say. Let \( n \) be an integer with \( n \geq p \) and let \( \{v_1, v_2, \ldots, v_n\} \) be the vertex set of a complete graph \( K_n \). Define an edge coloring \( c \) of \( K_n \) by \( c(v_i v_j) = i \) if \( i < j \). Hence \( c \) is a minimum edge coloring of \( K_n \). If \( k \) is the minimum positive integer such that \( v_k \) belongs to \( C \), then two edges of \( C \) are colored \( k \), implying that there is no rainbow \( F \) in \( K_n \). Since any other edge in \( C \) is not colored \( k \), it follows that \( F \) is not monochromatic either. Thus \( RR(F) \) is not defined.

For the converse, suppose that \( F \) is a forest of order \( p \geq 2 \). By known fact there exists and integer \( n \geq p \) such that for any edge coloring of \( K_n \) with positive integers, there is a complete subgraph \( G \) of order \( p \) in \( K_n \) that is either monochromatic or rainbow or has minimum or maximum coloring. If \( G \) is monochromatic or rainbow, then \( K_n \) contains a monochromatic or rainbow \( F \). Hence we may assume that the edge coloring of \( G \) is minimum or maximum, say the former. We show in this case that \( G \) contains a rainbow \( F \). If \( F \) is not a tree, then we can add edges to \( F \) to produce a tree \( T \) of order \( p \). Let \( V(G) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\} \), where \( i_1 < i_2 < \ldots < i_p \). Select some vertex \( v = v_{i_p} \) of \( T \) and label the vertices of \( T \) in the order \( v = v_{i_p}, v_{i_{p-1}}, \ldots, v_{i_2}, v_{i_1} \) of non decreasing distance from \( v \); that is,
\[
d(v_{i_j}, v) \geq d(v_{i_{j+1}}, v)
\]
for every integer \( j \) with \( 1 \leq j \leq p - 1 \). Hence there exists exactly on edge of \( T \) having color \( i_j \) for each \( j \) with \( 1 \leq j \leq p - 1 \). Thus \( T \) and hence \( F \) is rainbow. The rainbow Ramsey number \( RR(F) \) is therefore defined.

2.3. **Example:** For each integer \( k \geq 2 \), \( RR(K_{1,k}) = (k - 1)^2 + 2 \).

**Proof**

We first show that \( RR(K_{1,k}) \geq (k - 1)^2 + 2 \). Let \( n = (k - 1)^2 + 1 \). We consider two cases.
Case 1. k is odd. Then n is odd. Factor $K_n$ into $n^{1/2} = (k-1)^{1/2}$ Hamiltonian cycles each. Partition these cycles into $k-1$ sets $S_i (1 \leq i \leq k-1)$ of $k^{1/2}$ Hamiltonian cycles each. Color each edge of each cycle in $S_i$ with color $i$. Then there is neither a monochromatic $K_{1,k}$ nor a rainbow $K_{1,k}$.

Case 2. k is even. Then n is even. Factor $K_n$ into $n-1 = (k-1)^2$ 1-factors. Partition these 1-factors into $k-1$ sets $S_i (1 \leq i \leq k-1)$ of $k-1$ 1-factors. Color each edge of each 1-factor in $S_i$ color with $i$. Then there is neither a monochromatic $K_{1,k}$ nor a rainbow $K_{1,k}$.

Therefore, $RR(K_{1,k}) \geq (k-1)^2 + 2$. It remains to show that $RR(K_{1,k}) \leq (k-1)^2 + 2$. Let $N = (k-1)^2 + 2$ and let there be given an edge coloring of $K_N$ from any set of colors. Suppose that no monochromatic $K_{1,k}$ results. Let $v$ be a vertex of $K_N$. Since $\deg v = N-1$ and there is no monochromatic $K_{1,k}$, at most $k$ edges incident with $v$ can be colored the same. Thus there are at least $\lceil N/k-1 \rceil = k$ edges incident with $v$ that are colored differently, producing a rainbow $K_{1,k}$.

More generally, for two nonempty graphs $F_1$ and $F_2$, the rainbow Ramsey number $RR(F_1,F_2)$ is defined as the smallest positive integer $n$ such that if each edge of $K_n$ is colored from any set of colors, then there is either a monochromatic $F_1$ or a rainbow $F_2$ defined for every pair $F_1, F_2$ of non empty graphs.

3. DEFINITION

If the partite sets $u$ & $w$ of a complete bi partite graph contain $s$ & $t$ vertices. Then this graph is denoted by $K_{s,t}$, the graph $K_{1,t}$ is called star.

3.1. Theorem: Let $F_1$ and $F_2$ be two graphs without isolated vertices. The rainbow Ramsey number $RR(F_1,F_2)$ is defined if and only if $F_1$ is a star or $F_2$ is a forest.

Proof. First, we show that $RR(F_1,F_2)$ exists only if $F_1$ is a star or $F_2$ is a forest. Suppose that $F_1$ is not a star and $F_2$ is not a forest. Let $G$ be a complete graph of some order $n$ such that $V(G) = \{v_1,v_2,\ldots,v_n\}$ and such that both $F_1$ and $F_2$ are subgraphs of $G$. Define an $(n-1)$-edge coloring on $G$ such that the edge $v_i v_j$ is assigned the color $i$ if $i < j$. Hence this coloring is a minimum edge coloring of $G$.

Let $G_1$ be any copy of $F_1$ in $G$ and let $a$ be the minimum integer such that $v_a$ is a vertex of $G_1$. Then every edge incident with $v_a$ is colored $a$. since $G_1$ is not a star, some edge of $G_1$ is not incident with $v_a$ and is therefore not colored $a$. Hence $G_1$ is not monochromatic. Next, let $G_2$ be any copy of $F_2$ in $G$. Since $G_2$ is not a forest, $G_2$ contains a cycle $C$. Let $b$ be the minimum integer such that $v_b$ is a vertex of $G_2$ belonging to $C$. Since the two edges of $C$ incident with $v_b$ are colored $b$ (and $G_2$ contains at least two edges colored $b$), $G_2$ is not a rainbow subgraph of $G$. Hence $RR(F_1,F_2)$ is not defined.

We now verify the converse. Let $F_1$ and $F_2$ be two graphs without isolated vertices such that either $F_1$ is a star or $F_2$ is a forest. We show that there exists a positive integer $n$ such that for every edge coloring of $K_n$, either a monochromatic $F_1$ or a rainbow $F_2$ results. Suppose that the order of $F_1$ is $s+1$ and the order of $F_2$ is
t + 1 for positive integers s and t. Hence $F_1 = K_{1,s}$. We now consider two cases, depending on whether $F_1$ is a star or $F_2$ is a forest. It is convenient to begin with the case where $F_2$ is a forest.

**Case 1.** $F_2$ is a forest. If $F_2$ is not a tree, then we may add edges to $F_2$ so that a tree $G_2$ results. If $F_2$ is a tree, then let $G_2 = F_2$. Furthermore, if $F_1$ is not complete, then we may add edges to $F_1$ so that a complete graph $G_1 = K_{s+1}$ results. If $F_1$ is complete, then let $G_1 = F_1$. Hence $G_1 = K_{s+1}$ and $G_2$ is a tree of order $t + 1$. We now show that $RR(G_1,G_2)$ is defined by establishing the existence of a positive integer $n$ such that any edge coloring of $K_n$ from any set of colors results in either a monochromatic $G_1$ or a rainbow $G_2$. This, in turn, implies the existence of monochromatic $F_1$ or a rainbow $F_2$. We now consider two cases, depending on whether $G_2$ is a star.

**Sub case 1.1.** $G_2$ is a star of order $t + 1$, that is, $G_2 = K_{1,t}$. Therefore, in this subcase, $G_1 = K_{s+1}$ and $G_2 = K_{1,t}$. (This subcase will aid us later in the project) In this subcase, let

$$n = \sum_{i=0}^{(s-1)(t-1)+1} (t-1)^i$$

and let an edge coloring of $K_n$ be given from any set of colors. If $K_n$ contains a vertex incident with $t$ or more edges assigned distinct colors, then $K_n$ contains a rainbow $G_2$. Hence we may assume that every vertex of $K_n$ is incident with at most $t-1$ edges assigned distinct colors. Let $v_1$ be a vertex of $K_n$. Since the degree of $v_1$ in $K_n$ is $n-1$, there are at least

$$\frac{n-1}{t-1} = \sum_{i=0}^{(s-1)(t-1)} (t-1)^i$$

edges incident with $u_1$ that are assigned the same color, say color $c_1$.

Let $S_1$ be the set of vertices joined to $v_1$ by edges colored $c_1$ and let $v_2 \in S_1$. There are at least

$$\frac{|S_1| - 1}{t-1} \geq \sum_{i=0}^{(s-1)(t-1)-1} (t-1)^i$$

edges of the same color, say color $c_2$, joining $v_2$ and vertices of $S_1$, where possibly $c_2 = c_1$. Let $S_2$ be the set of vertices in $S_1$ joined to $v_2$ by edges colored $c_2$. Continuing in this manner, we construct sets $S_1, S_2, \ldots, S_{(s-1)(t-1)}$ and vertices $v_1, v_2, \ldots, v_{(s-1)(t-1)+1}$ such that $2 \leq i \leq (s-1)(t-1)+1$, the vertex $v_i$ belongs to $S_{i-1}$ and is joined to at least

$$\frac{|S_1| - 1}{t-1} \geq \sum_{i=0}^{(s-1)(t-1)-1} (t-1)^i$$

vertices of $S_{i-1}$ by edges colored $c_i$. Finally, in the set $S_{(s-1)(t-1)}$, the vertex
v_{(s-1) (t-1)+1} is joined to a vertex v_{(s-1)(t-1)+2} in S_{(s-1)(t-1)} by an edge colored c_{(s-1)(t-1)+1}. Thus we have a sequence
\[ U_1, U_2, \ldots, V_{(s-1)(t-1)+2} \]
of vertices such that every edge \( v_i, v_j \) for \( 1 \leq i < j \leq (s-1) (t-1) + 2 \) is colored \( c_i \) and where the colors \( c_1, c_2, \ldots, c_{(s-1)(t-1)+1} \) are not necessarily distinct. In the complete subgraph \( H \) of order \( (s-1) (t-1) + 2 \) induced by the vertices listed in (11.3), the vertex \( v_{(s-1)(t-1)+2} \) is incident with at most \( t-1 \) edges having distinct colors. Hence there is a set of at least
\[ \left[ \frac{(s-1)(t-1) + 1}{t-1} \right] = s \]
Vertices in \( H \) joined to \( v_{(s-1)(t-1)+2} \) by edges of the same color. Let \( v_{i_1}, v_{i_2}, \ldots, v_{i_s} \) be \( s \) of these vertices, where \( i_1 < i_2 < \ldots < i_s \). Then \( c_{i_1} = c_{i_2} = \ldots = c_{i_s} \), and the complete subgraph of order \( s+1 \) induced by
\{ \( v_{i_1}, v_{i_2}, \ldots, v_{i_s}, v_{(s-1)(t-1)+2} \) \}
is monochromatic.

**Subcase 1.2** \( G_2 \) is a tree of order \( t+1 \) that is not necessarily a star. Recall that \( G_1 = K_{s+1} \). We proceed by induction on the positive integer \( t \). If \( t = 1 \) or \( t = 2 \), then \( G_2 \) is a star and the base case of the induction follows by subcase 1.1. Suppose that \( RR(G_1, G_2) \) exists for \( G_1 = K_{s+1} \) and for every tree \( G_2 \) of order \( t + 1 \) where \( t \geq 2 \). Let \( T \) be a tree of order \( t + 2 \). We show that \( RR(G_1, T) \) exists. Let \( v \) be an end-vertex of \( T \) and let \( v \) be the vertex of \( T \) that is adjacent to \( v \). Let \( T' = T - v \). Since \( T' \) is a tree of order \( t + 1 \), it follows by the induction hypothesis that \( RR(G_1, T') \) exists, say \( RR(G_1, T') = p \). Hence for any edge coloring of \( K_p \) from any set of colors, there is either a monochromatic \( G_1 = K_{s+1} \) or a rainbow \( T' \). From sub case 1.1, we know that \( RR(G_1, K_{1,t+1}) \) exists. Suppose that \( RR(G_1, K_{1,t+1}) = q \) and let \( n = pq \) in this subcase.

Let there be given an edge coloring of \( K_n \) using any number of colors. Consider a partition of the vertex set of \( K_n \) into \( q \) mutually disjoint sets of \( p \) vertices each. By the induction hypothesis, the complete subgraph induced by each set of \( p \) vertices contains either a monochromatic \( K_{s+1} \) or rainbow \( T' \). If a monochromatic \( K_{s+1} \) occurs in any of these complete subgraph \( K_p \), then subcase 1.2 is verified. Hence we may assume that there are \( q \) pair wise mutually rainbow copies.

\[ T_1, T_2, \ldots, T_q \]
of \( T' \), where \( u_i \) is the vertex in \( T_i \) \((1 \leq i \leq q)\) corresponding to the vertex \( u \) in \( T \).
Let \( H \) be the complete subgraph of order \( q \) induced by \( \{ u_1, u_2, \ldots, u_q \} \). Since \( RR(K_{s+1}, K_{1,t+1}) = q \), it follows that either \( H \) contains a monochromatic \( K_{s+1} \) or a rainbow \( K_{1,t+1} \). If \( H \) contains a monochromatic \( K_{s+1} \), then once again, the proof of subcase 1.2 is complete. So we may assume that \( H \) contains a rainbow \( K_{1,t+1} \). Let \( u_j \) be the center of a rainbow star \( K_{1,t+1} \) in \( H \). At least one of the \( t + 1 \) colors of the edges of \( K_{1,t+1} \) is different from the colors of the \( t \) edges of \( T_j \). Adding the edge having this color at \( u_j \) in \( T_j \) produces a rainbow copy of \( T \).
Case 2. \(F_1\) is a star. Denote \(F_1\) by \(G_1\) as well and so \(G_1 = K_{1,s}\). If \(F_2\) is complete, then let \(G_2 = F_2\). If \(F_2\) is not complete, then we may add edges to \(F_2\) so that a complete graph \(G_2 = K_{t+1}\) results. We verify that \(RR(G_1, G_2)\) exists by establishing the existence of a positive integer \(n\) such that for any edge coloring of \(K_n\) from any set of colors, either a monochromatic \(G_1\) or a rainbow \(G_2\) results. This then shows that \(K_n\) will have a monochromatic \(F_1\) or a rainbow \(F_2\). For positive integers \(p\) and \(r\) with \(r < p\), let

\[
p^r = \frac{p!}{(p-r)^{r!}} = p(p-1) \cdots (p-r+1).
\]

Now let \(n\) be an integer such that \(s-1\) divides \(n-1\) and

\[
n \geq 3 + \frac{(s-1)(t+2)^4}{8}
\]

Then \(n-1 = (s-1)q\) for some positive integer \(q\). Let there be given an edge coloring of \(K_n\) from any set of colors and suppose that no monochromatic \(G_1 = K_{1,s}\) occurs. We show that there is a rainbow \(G_2 = K_{t+1}\). Observe that the total number of different copies of \(K_{t+1}\) in \(K_n\) is implying the existence of at least one rainbow \(K_{t+1}\).

First consider the number of copies of \(K_{t+1}\) containing adjacent edges \(uv\) and \(uw\) that are colored the same. There are \(n\) possible choice for the vertex \(u\). Suppose that there are \(a_i\) edges incident with \(u\) that are colored \(i\) for \(1 \leq i \leq k\). Then

\[
\sum_{i=1}^{k} a_i = n - 1,
\]

Where, by assumption, \(1 \leq a_i \leq s - 1\) for each \(i\). For each color \(i(1 \leq i \leq k)\), the number of different choices for \(v\) and \(w\) where \(uv\) and \(uw\) are colored \(i\) is \(\binom{a_i}{2}\). Hence the number of different choices for \(u\) and \(w\) where \(uv\) and \(uw\) are colored the same is

\[
\sum_{i=1}^{k} \binom{a_i}{2}
\]

since the maximum value of this sum occurs when each \(a_i\) is as large as possible, the largest value of this sum is when each \(a_i\) is \(s - 1\) and when \(k = q\), that is, there are at most

\[
\sum_{i=1}^{q} \binom{s-1}{2} = q \binom{s-1}{2}
\]
choices for \(v\) and \(w\) such that \(uv\) and \(uw\) are colored the same. Since there are \(\binom{n-3}{t-2}\) choices for the remaining \(t-2\) vertices of \(K_{t+1}\), it follows that there are at most
\[
n_q \left( \frac{s-1}{2} \right) \binom{n-3}{t-2}
\]
copies of \(K_{t+1}\) containing two adjacent edges that are colored the same.

We now consider copies of \(K_{t+1}\) in which there are two nonadjacent edges, say \(e = xy\) and \(f = wz\), colored the same. There are \(\binom{n}{2}\) choices for \(e\) and \(n - 2\) choices for one vertex, say \(w\), that is incident with \(f\). The vertex \(w\) is incident with at most \(s - 1\) edges having the same color as \(e\) and not adjacent to \(e\). Since there are four ways of counting such a pair of edges in this way (namely \(e\) and either \(w\) or \(z\), or \(f\) and either \(x\) or \(y\)), there are at most
\[
\frac{\binom{n}{2}(n-2)(s-1)}{4} = \frac{n(n-1)(n-2)(s-1)}{8}
\]
Ways to choose nonadjacent edges of the same color and \(\binom{n-t}{i-3}\) ways to choose the remaining \(t-3\) vertices of \(K_{t+1}\). Hence there are at most
\[
\frac{n(n-1)(n-2)(s-1)}{8} \binom{n-4}{t-3}
\]
Copies of \(K_{t+1}\) containing two nonadjacent edges that are colored the same. Therefore, the number of non-rainbow copies of \(K_{t+1}\) is at most
\[
n_q \left( \frac{s-1}{2} \right) \binom{n-3}{t-2} + \frac{n(n-1)(n-2)(s-1)}{8} \binom{n-4}{t-3}
\]
\[
= \frac{n}{t+1} \left[ \frac{(s-2)(t+1)^{(3)}}{2(n-2)} + \frac{(s-1)(t+1)^{(4)}}{8(n-3)} \right]
\]
\[
< \frac{n}{t+1} \left[ \frac{(s-1)(t+1)^{(3)}}{2(n-3)} + \frac{(s-1)(t+1)^{(4)}}{8(n-3)} \right]
\]
\[
= \frac{n}{t+1} \left[ \frac{(s-1)(t+1)^{(3)}(t+2)}{8(n-3)} \right]
\]
\[
= \frac{n(t+1)}{8(n-3)} \leq \frac{n}{t+1},
\]
Where the final inequality follows from known theorem, the rainbow Ramsey number is defined if and only if $F$ is a forest hence there is a rainbow $K_{t+1}$ in $K_n$.

4. REFERENCES