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RAMANUJAN'S SPT-CRANK FOR MARKED OVERPARTITIONS

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ABSTRACT

In 1916, Ramanujan's showed the spt-crank for marked overpartitions. The corresponding special functions $\overline{S}(z,x)$, $\overline{S}_1(z,x)$ and $\overline{S}_2(z,x)$ are found in Ramanujan's notebooks, part 111. In 2009, Bingmann, Lovejoy and Osburn defined the generating functions for $\overline{spt}(n)$, $\overline{spt}_1(n)$ and $\overline{spt}_2(n)$. In 2012, Andrews, Garvan, and Liang defined the $\overline{sptcrank}$ in terms of partition pairs. In this article the number of smallest parts in the overpartitions of n with smallest part not overlined, not overlined and odd, not overlined and even are discussed, and the vector partitions and $\overline{S}$ - partitions with 4 components, each a partition with certain restrictions are also discussed. The generating functions $\overline{spt}(n)$, $\overline{spt}_1(n)$, $\overline{spt}_2(n)$, $M_{\overline{S}}(m,n)$, $M_{\overline{S}_1}(m,n)$ $M_{\overline{S}_2}(m,n)$ are shown with the corresponding results in terms of modulo 3, where the generating functions $M_{\overline{S}}(m,n)$, $M_{\overline{S}_1}(m,n)$ $M_{\overline{S}_2}(m,n)$ are collected from Ramanujan's notebooks, part 111. This paper shows how to prove the Theorem 1 in terms of $M_{\overline{S}}(m,n)$, Theorem 2 in terms of $M_{\overline{S}_1}(m,n)$ and Theorem 3 in terms of $M_{\overline{S}_2}(m,n)$ respectively with the numerical examples, and shows how to prove the Theorems 4,5 and 6 with the help of $\overline{sptcrank}$ in terms of partition pairs. In 2014, Garvan and Jennings-Shaffer are able to defined the $\overline{sptcrank}$ for marked overpartitions. This paper also shows another results with the help of 6 $\overline{SP}$ -partition pairs of 3, help of 20 $\overline{SP}_1$-partition pairs of 5 and help of 15 $\overline{SP}_2$ -partition pairs of 8 respectively.

Keywords:
Components, congruent, crank, overpartitions, overlined, weight.

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1. INTRODUCTION

In this paper we give some related definitions of $\overline{\text{spt}}(n)$, $\overline{spt_1}(n)$, $\overline{spt_2}(n)$, various product notations, vector partitions and $\overline{S}$ - partitions, $M_{\overline{S}}(m,n)$, $M_{\overline{S}}(m,t,n)$, $M_{\overline{S}_1}(m,n)$, $M_{\overline{S}_1}(m,t,n)$, $M_{\overline{S}_2}(m,n)$, $M_{\overline{S}_2}(m,t,n)$, $\overline{S}\ (z,x), \overline{S}_1(z,x), \overline{S}_2(z,x)$, marked partition and $\overline{\text{sptcrank}}$ for marked overpartitions. We discuss the generating functions for $\overline{\text{spt}}(n)$, $\overline{spt_1}(n)$, $\overline{spt_2}(n)$ and prove the Corollaries 1, 2 and 3 with the help of generating functions for $M_{\overline{S}}(m,n)$, $M_{\overline{S}_1}(m,n)$ and $M_{\overline{S}_2}(m,n)$ respectively. We prove the Results 1, 2 and 3 with the help of 8 vector partitions from $\overline{S}$ of 3, from $\overline{S}_1$ of 3 and of 3 vector partitions from $\overline{S}_2$ of 4 respectively. We prove the Theorems 1, 2 and 3 with the help of various generating functions and establish the Corollaries 4, 5 and 6 with the help special series $\overline{S}\ (z,x)$, $\overline{S}_1(z,x)$ and $\overline{S}_2(z,x)$ respectively, where the special series $\overline{S}\ (z,x)$, $\overline{S}_1(z,x)$ and $\overline{S}_2(z,x)$ are collected from **Ramanujan's notebooks, part *111*** , and prove the Theorems 4, 5 and 6 with the help of $\overline{sptcrank}$ in terms of partition pairs (λ_1,λ_2) when $0 < \text{s}(\lambda_1) \le \text{s}(\lambda_2)$. We establish the Results 4, 5 and 6 using the $\overline{crank}$ of partition pairs $\vec{\lambda} = (\lambda_1,\lambda_2)$ and analyze the Corollaries 7,8, and 9 with the help of 42 marked overpartitions of 6, 36 marked overpartitions of 6, 260 marked overpartitions of 10 respectively.

2. SOME RELATED DEFINITIONS

$\overline{spt}(n)$: The number of smallest parts in the overpartitons of *n* with smallest part not overlined is denoted by $\overline{spt}(n)$ for example; $\overline{spt}(1)=1$, $\overline{spt}(2)=3$, $\overline{spt}(3)=6$, $\overline{spt}(4)=13$ ...

$\overline{spt_1}(n)$**:** The number of smallest parts in the overpartitons of *n* with smallest part not overlined and odd is denoted by $\overline{spt_1}(n)$, for example; $\overline{spt_1}(1)=1$, $\overline{spt_1}(2)=2$, $\overline{spt_1}(3)=6$, $\overline{spt_1}(4)=10$ ...

$\overline{spt_2}(n)$**:** The number of smallest parts in the overpartitions of *n* with smallest part not overlined and even is denoted by $\overline{spt_2}(n)$ for example; $\overline{spt_2}(1)=0$, $\overline{spt_2}(2)=1$, $\overline{spt_2}(3)=0$, $\overline{spt_2}(4)=3$ ...

Product Notations:

$(x)_\infty = (1-x)(1-x^2)(1-x^3)...$

$(x^2;x^2)_\infty = (1-x^2)(1-x^4)...$

$(x)_k = (1-x)(1-x^2)(1-x^3)....(1-x^k)$

$(-x^5;x)_\infty = (1+x^5)(1+x^6)(1+x^7)...$

Where $|x|<1$.

Vector Partitions and $\overline{\text{S}}$ - partitions [5]:

A vector partition can be done with 4 components each partition with certain restrictions.

Let $\vec{V} = D \times P \times P \times D$ where D denotes the set of all partitions into distinct parts, P denotes the set of all partitions. For a partition π, we let $s(\pi)$ denote the smallest part of π (with the convention that the empty partition has smallest part ∞), $\#(\pi)$ the number of parts in π, and $|\pi|$ the sum of the parts of π.

For $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4) \in \vec{V}$, we define the weight $\omega(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$, the crank $c(\vec{\pi}) = \#(\pi_2) - \#(\pi_3)$, the norm $|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3| + |\pi_4|$.

We say $\vec{\pi}$ is a vector partition of n if $|\vec{\pi}| = n$. Let $\bar{S}$ denote the subset of $\bar{V}$ and it is given by

$$\bar{S} = \left\{ (\pi_1, \pi_2, \pi_3, \pi_4) \in \vec{V}, 1 \le s(\pi_1) < \infty, s(\pi_1) \le s(\pi_2), s(\pi_1) \le s(\pi_3), s(\pi_1) < s(\pi_4) \right\}.$$

$M_{\bar{S}}(m,n)$**:** The number of vector partitions of n in $\bar{S}$ with crank m counted according to the weight ω is exactly $M_{\bar{S}}(m,n)$.

$M_{\bar{S}}(m,t,n)$**:** The number of vector partitions of n in $\bar{S}$ with crank congruent to m modulo t counted according to the weight ω is exactly $M_{\bar{S}}(m,t,n)$.

Let $\overline{S_1}$ denote the subset of $\bar{S}$ with $s(\pi_1)$ odd.

$M_{\bar{S}_1}(m,n)$: The number of vector partitions of n in $\bar{S}_1$ with crank m counted according to the weight ω is exactly $M_{\bar{S}_1}(m,n)$.

$M_{\bar{S}_1}(m,t,n)$: The number of vector partitions of n in $\bar{S}_1$ with crank congruent to m modulo t counted according to the weight ω is exactly $M_{\bar{S}_1}(m,t,n)$.

Let $\overline{S_2}$ denotes the subset of $\bar{S}$ with $s(\pi_1)$ even.

$M_{\bar{S}_2}(m,n)$: The number of vector partitions of *n* in $\bar{S}_2$ with crank *m* counted according to the weight ω is exactly $M_{\bar{S}_2}(m,n)$.

$M_{\bar{S}_2}(m,t,n)$: The number of vector partitions of *n* in $\bar{S}_2$ with crank congruent to *m* modulo *t* counted according to the weight ω is exactly $M_{\bar{S}_2}(m,t,n)$.

$\bar{S}(z,x)$: The series $\bar{S}(z,x)$ is defined by the generating function for $M_{\bar{S}}(m,n)$

$$i.e., \bar{S}(z,x) = \sum_{n=1}^{\infty} \frac{x^n(-x^{n+1};x)_\infty (x^{n+1};x)_\infty}{(zx^n;x)_\infty (z^{-1}x^n;x)_\infty} = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\bar{S}}(m,n).$$

$\bar{S}_1(z,x)$: The series $\bar{S}_1(z,x)$ is defined by the generating function for $M_{\bar{S}_1}(m,n)$;

$$\bar{S}_1(z,x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}(-x^{2n+2};x)_\infty (x^{2n+2};x)_\infty}{(zx^{2n+1};x)_\infty (z^{-1}x^{2n+1};x)_\infty} = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\bar{S}_1}(m,n) z^m x^n.$$

$\bar{S}_2(z,x)$: The series $\bar{S}_2(z,x)$ is defined by the generating function for $M_{\bar{S}_2}(m,n)$

i.e., $\overline{S}_2(z, x) = \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} M_{\overline{S}_2}(m,n)z^m x^n$.

Marked Partition [1]: We define a marked partition as a pair (λ, k) where λ is a partition and k is an integer identifying one of its smallest parts i.e. k =1, 2, ... $\nu(\lambda)$, where $\nu(\lambda)$ is the number of smallest parts of λ.

$\overline{\text{spt}}$crank**for Marked overpartitions[6]**: We define a marked overpartitions of n as a pair (π, j) where π is an overpartition of n in which the smallest parts is not overlined and j is an integer $1 \le j \le \nu(\pi)$, where $\nu(\pi)$ is the number of smallest parts to π. It is clear that $\overline{\text{spt}}(n)$ = # of marked overpartitions (π, j) of *n*.

For example; there are 3 marked overpartitions of 2

Like- (2,1),(1+1,1),(1+1,2) so that $\overline{\text{spt}}(2)$=3.

Again there are 6 marked overpartitions of 3 like- (3,1), (2+1,1), ($\overline{2}$ +1,1), (1+1+1,1) (1+1+1,2) and (1+1+1,3) so that $\overline{\text{spt}}(3)$ = 6.

3. THE GENERATING FUNCTION FOR $\overline{\text{spt}}(n)$

$\overline{spt}$ (n) is the number of smallest parts in the overpartitons of *n* with smallest part not overlined like-

Table-1

N	The type of smallest parts in the overpartitons of *n*	$\overline{spt}(n)$
1	$\dot{1}$	1
2	$\dot{2}$, $\dot{1}+\dot{1}$	3
3	$\dot{3}$, $2+\dot{1}$, $\overline{2}+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}$	6
4	$\dot{4}$, $3+\dot{1}$, $\overline{3}+1$, $\dot{2}+\dot{2}$, $2+\dot{1}+\dot{1}$, $\overline{2}+\dot{1}+\dot{1}+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}+\dot{1}$	13
...	...	...

We make the expression

$$\overline{spt}(1)x + \overline{spt}(2)x^2 + \overline{spt}(3)x^3 + \overline{spt}(4)x^4 + ...$$

$$= 1.x + 3.x^2 + 6.x^3 + 13.x^4 + 22.x^5 + 42.x^6 + ...$$

$$= \frac{x(1+x^2)(1+x^3).....}{(1-x)^2(1-x^2)(1-x^3)......} + \frac{x^2(1+x^3)(1+x^4).....}{(1-x^2)^2(1-x^3)(1-x^4)......} + \ldots \quad \text{[Andrews et al (2013)]}$$

$$= \frac{x\ (-x^2;x)_\infty}{(1-x)^2(x^2;x)_\infty} + \frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(x^3;x)_\infty} + ...$$

$$= \sum_{n=1}^{\infty} \frac{x^n(-x^{n+1};x)_\infty}{(1-x^n)^2(x^{n+1};x)_\infty}$$

$$\therefore \sum_{n=1}^{\infty} \frac{x^n(-x^{n+1};x)_\infty}{(1-x^n)^2(x^{n+1};x)_\infty} = \sum_{n=1}^{\infty} \overline{spt}(n)x^n.$$

From above we get; $\overline{\text{spt}}(3) = 6$, $\overline{\text{spt}}(6) = 42$,....

i.e. $\overline{\text{spt}}(3.1) = 6 \equiv 0 \pmod 3$, $\overline{\text{spt}}(3.2) = 42 \equiv 0 \pmod 3$, ...

We can conclude that;

$\overline{spt}(3n) \equiv 0$ (mod 3), for $n \geq o$.

3.1. THE GENERATING FUNCTION FOR $\overline{spt_1}(n)$

$\overline{spt_1}(n)$ is the number of smallest parts in the overpartitons of *n* with smallest part not overlined and odd like-

Table-2

n	The type of smallest parts in the overpartitions of n	$\overline{spt_1}(n)$
1	$\dot{1}$	1
2	$\dot{1}+\dot{1}$	2
3	$\dot{3},\ 2+\dot{1},\ \overline{2}+\dot{1},\ \dot{1}+\dot{1}+\dot{1}$	6
4	$3+\dot{1},\ \overline{3}+\dot{1},\ 2+\dot{1}+\dot{1},\ \overline{2}+\dot{1}+\dot{1},\ \dot{1}+\dot{1}+\dot{1}+\dot{1}$	10
…	….	…

We make the expression

$$\overline{spt_1}(1)x+\overline{spt_1}(2)x^2+\overline{spt_1}(3)x^3+\overline{spt_1}(4)x^4+\ldots$$

$$=x^2+2x^2+6x^3+10.x^4+20x^5+36.x^6+\ldots$$

$$=\frac{x(-x^2;x)_\infty}{(1-x)^2(x^2;x)_\infty}+\frac{x^3(-x^4;x)_\infty}{(1-x^3)^2(x^4;x)_\infty}+\frac{x^5(-x^6;x)_\infty}{(1-x^5)^2(x^6;x)_\infty}+..\text{[Lovejoy et al (2009)]}$$

$$=\sum_{n=0}^{\infty}\frac{x^{2n+1}(-x^{2n+2};x)_\infty}{(1-x^{2n+1})^2(x^{2n+2};x)_\infty}$$

$$\therefore\sum_{n=0}^{\infty}\frac{x^{2n+1}(-x^{2n+2};x)_\infty}{(1-x^{2n+1})^2(x^{2n+2};x)_\infty}=\sum_{n=1}^{\infty}\overline{spt_1}(n)x^n.$$

From above we get $\overline{spt_1}(3)=6,\ \overline{spt_1}(6)=36,\ldots$

i.e., $\overline{spt_1}(3.1)=6\equiv 0$ (mod 3), $\overline{spt_1}(3.2)=36\equiv 0$ (mod 3), ...

We can conclude that

$\overline{spt_1}(3n)\equiv 0$ (mod 3), for $n \geq o$ ([4])

From above we get $\overline{spt_1}(1)=1,\ \overline{spt_1}(9)=165,\ldots$

i.e. $\overline{spt_1}(1)=1\equiv 1$ (mod 2), $\overline{spt_1}(3^2)=165\equiv 1$ (mod 2), ...

We can conclude that

$\overline{spt_1}(n)\equiv 1$ (mod 2), if n is an odd square.

Again we get;

$\overline{spt_1}(5)=10,\ \overline{spt_1}(10)=260,\ldots$

i.e. $\overline{spt_1}(5.1)=10\equiv 0$ (mod 5), $\overline{spt_1}(5.2)=260\equiv 0$ (mod 5), ...

We can conclude that;

$\overline{spt_1}(5n)\equiv 0$ (mod 5).

3.2. THE GENERATING FUNCTION FOR $\overline{spt_2}$ (*n*):

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$\overline{spt_2}(n)$ is the number of smallest parts in the overpartitions of *n* with smallest part not overlined and even like-

Table-3

n	The type of smallest parts in the overpartitions of n	$\overline{spt_2}(n)$
1	none	0
2	$\dot{2}$	1
3	none	0
4	$\dot{4}$, $\dot{2}+\dot{2}$	3
5	$3+\dot{2}$, $\bar{3}+\dot{2}$	2
…	…	…

We make the expression

$$\overline{spt_2}(1)x+\overline{spt_2}(2)x^2+\overline{spt_2}(3)x^3+\overline{spt_2}(4)x^2+\overline{spt_2}(5)x^2+\ldots$$

$$=o.x+1.x^2+o.x^3+3.x^4+2.x^5+6.x^6+\ldots$$

$$=\frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(x^3;x)_\infty}+\frac{x^4(-x^5;x)_\infty}{(1-x^4)^2(x^5;x)_\infty}+\ldots \text{ [10]}$$

$$=\sum_{n=1}^{\infty}\frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$$

$$\therefore \sum_{n=1}^{\infty}\frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}=\sum_{n=1}^{\infty}\overline{spt_2}(n)x^n.$$

From above we get $\overline{spt_2}(3)=0,\ \overline{spt_2}(6)=6,\ldots$

i.e., $\overline{spt_2}(3.1)=0\equiv 0$ (mod 3), $\overline{spt_2}(3.2)=6\equiv 0$ (mod 3), ...

We can conclude that $\overline{spt_2}(3n)\equiv 0(\text{mod}\,3)$.

We also get $\overline{spt_2}(4)=3,\ \overline{spt_2}(7)=6,\ldots$

i.e., $\overline{spt_2}(3+1)=3\equiv 0(\text{mod}\,3),\ \overline{spt_2}(3.2+1)=6\equiv 0(\text{mod}\,3),\ \ldots$

We can conclude that $\overline{spt_2}(3n+1)\equiv 0(\text{mod}\,3)$.

Again from above we get; $\overline{spt_2}(3)=0,\ \overline{spt_2}(8)=15,\ldots$

i.e., $\overline{spt_2}(3)=0\equiv 0(\text{mod}\,5),\ \overline{spt_2}(5+3)=15\equiv 0(\text{mod}\,5)$, ...

We can conclude that $\overline{spt_2}(5n+3)\equiv 0(\text{mod}\,5)$. [5]

Corollary 1: $\overline{spt}\,(n)=\sum_{m=-\infty}^{\infty}M_{\bar{S}}(m,n)$

Proof: The generating function for $M_{\bar{S}}(m,n)$ [3] is given by;

$$\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}M_{\bar{S}}(m,n)\,z^m x^n=\sum_{n=1}^{\infty}\frac{x^n(x^{n+1};x)_\infty(-x^{n+1};x)_\infty}{(zx^n;x)_\infty(z^{-1}x^n;x)_\infty}$$

If z = 1, then we get; $\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} M_{\bar{S}}(m,n)\, x^n = \sum_{n=1}^{\infty} \frac{x^n (x^{n+1};x)_\infty (-x^{n+1};x)_\infty}{(x^n;x)_\infty (x^n;x)_\infty}$

$$= \frac{x(x^2;x)_\infty(-x^2;x)_\infty}{(x;x)_\infty^2} + \frac{x^2(x^3;x)_\infty(-x^4;x)_\infty}{(x^2;x)^2{}_\infty} + \ldots$$

$$= \frac{x(-x^2;x)_\infty(1-x^2)(1-x^3)\ldots}{(1-x)^2(1-x^2)^2\left(1-x^3\right)^2\ldots} + \frac{x^2(-x^3;x)_\infty(1-x^3)(1-x^4)\ldots}{(1-x^2)^2(1-x^3)^2\left(1-x^4\right)^2\ldots} + \ldots$$

$$= \frac{x(-x^2;x)_\infty}{(1-x)^2(1-x^2)\left(1-x^3\right)\ldots} + \frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(1-x^3)(1-x^4)\ldots} + \ldots$$

$$= \frac{x(-x^2;x)_\infty}{(1-x)^2\left(x^2;x\right)_\infty} + \frac{x^2(-x^3;x)_\infty}{(1-x^2)^2\left(x^3;x\right)_\infty} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{x^n(-x^{n+1};x)_\infty}{(1-x^n)^2(x^{n+1};x)_\infty} = \sum_{n=1}^{\infty} \overline{spt}\,(n)x^n .$$

i.e; $\sum_{n=1}^{\infty} \overline{spt}\,(n)x^n = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} M_{\bar{S}}(m,n)\, x^n$.

Now equating the co-efficient of x^n from both sides we get;

$\overline{spt}\,(n) = \sum_{m=-\infty}^{\infty} M_{\bar{S}}(m,n)$. Hence The Corollary.

Corollary 2: $\overline{spt_1}(n) = \sum_{m=-\infty}^{\infty} N_{\bar{S}_1}(m,n)$

Proof: we get $\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N_{\bar{S}_1}(m,n)\, z^m x^n = \sum_{n=0}^{\infty} \frac{x^{2n+1}(-x^{2n+2};x)_\infty (x^{2n+2};x)_\infty}{(zx^{2n+1};x)_\infty (z^{-1}x^{2n+1};x)_\infty}$.

If z = 1, then we get; $\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N_{\bar{S}_1}(m,n)\, x^n = \sum_{n=0}^{\infty} \frac{x^{2n+1}(-x^{2n+2};x)_\infty (x^{2n+2};x)_\infty}{(x^{2n+1};x)_\infty (x^{2n+1};x)_\infty}$

$$= \frac{x(-x^2;x)_\infty(x^2;x)_\infty}{(x;x)_\infty^2} + \frac{x^3(-x^4;x)_\infty(x^4;x)_\infty}{(x^3;x)^2{}_\infty} + \ldots$$

$$= \frac{x(-x^2;x)_\infty(1-x^2)(1-x^3)\ldots}{(1-x)^2(1-x^2)^2\ldots} + \frac{x^3(-x^4;x)_\infty(1-x^4)(1-x^5)\ldots}{(1-x^3)^2(1-x^4)^2\ldots} + \ldots$$

$$= \frac{x(-x^2;x)_\infty}{(1-x)^2(1-x^2)\ldots} + \frac{x^3(-x^4;x)_\infty}{(1-x^3)^2(1-x^4)(1-x^5)\ldots} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}(-x^{2n+2};x)_\infty}{(1-x^{2n+1})^2(x^{2n+2};x)_\infty} = \sum_{n=1}^{\infty} \overline{spt_1}(n)x^n .$$

i.e., $\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N_{\bar{S}_1}(m,n)\, x^n = \sum_{n=1}^{\infty} \overline{spt_1}(n) x^n$.

Now equating the co-efficient of x^n from both sides we get

$\overline{spt_1}(n) = \sum_{m=-\infty}^{\infty} N_{\bar{S}_1}(m,n)$. Hence The Corollary.

Corollary 3: $\overline{spt_2}(n) = \sum_{m=-\infty}^{\infty} M_{\bar{S}_2}(m,n)$

Proof: The generating function for $M_{\bar{S}_2}(m,n)$ is given by;

$$\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\bar{S}_2}(m,n)\, z^m x^n = \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.$$

If $z = 1$, then, $\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\bar{S}_2}(m,n)\, \mathrm{x}^{\mathrm{n}} = \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(x^{2n};x)_\infty(x^{2n};x)_\infty}$

$$= \frac{x^2(-x^3;x)_\infty(x^3;x)_\infty}{(x^2;x)_\infty^2} + \frac{x^4(-x^5;x)_\infty(x^5;x)_\infty}{(x^4;x)^2{}_\infty} + \ldots$$

$$= \frac{x^2(-x^3;x)_\infty(1-x^3)(1-x^4)\ldots}{(1-x^2)^2(1-x^3)^2\ldots} + \frac{x^4(-x^5;x)_\infty(1-x^5)(1-x^6)\ldots}{(1-x^4)^2(1-x^5)^2\ldots} + \ldots$$

$$= \frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(1-x^3)(1-x^4)\ldots} + \frac{x^4(-x^5;x)_\infty}{(1-x^4)^2(1-x^5)(1-x^6)\ldots} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty} = \sum_{n=1}^{\infty} \overline{spt_2}(n) x^n.$$

i.e., $\sum_{n=1}^{\infty} \overline{spt_2}(n) x^n = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\bar{S}_2}(m,n)\, x^n$.

Now equating the co-efficient of x^n from both sides we get;

$\overline{spt_2}(n) = \sum_{m=-\infty}^{\infty} M_{\bar{S}_2}(m,n)$. Hence The Corollary.

Result 1: $M_{\bar{S}}(0,3,3) = M_{\bar{S}}(1,3,3) = M_{\bar{S}}(2,3,3) = \frac{1}{3}\overline{spt}(3)$.

Proof: We prove the result with an example. We see the vector partitions from $\bar{S}$ of 3 along with their weights and cranks are given as follows.

Table 4:

$\bar{S}$-vector partition $(\vec{\pi})$ of 3	Weight $\omega(\vec{\pi})$	Crank $c(\vec{\pi})$

$\vec{\pi}_1 = (1,\phi,\phi,2)$	+ 1	0
$\vec{\pi}_2 = (1,\phi,1+1,\phi)$	+1	-2
$\vec{\pi}_3 = (1,1+1,\phi,\phi)$	+1	2
$\vec{\pi}_4 = (1, 1, 1,\phi)$	+1	0
$\vec{\pi}_5 = (1,\phi,2,\phi)$	+1	-1
$\vec{\pi}_6 = (1,2,\phi,\phi)$	+1	1
$\vec{\pi}_7 = (1+2,\phi,\phi,\phi)$	-1	0
$\vec{\pi}_8 = (3,\phi,\phi,\phi)$	+1	0
	$\sum \omega(\vec{\pi}) = 6$	

Here we have used ϕ to indicate the empty partition.

Thus we have, $M_{\bar{S}}(0,3,3) = +1+1-1+1 = 2$, $M_{\bar{S}}(1,3,3) = M_{\bar{S}}(-2,3,3) = 1+1 = 2$,
$M_{\bar{S}}(2,3,3) = M_{\bar{S}}(-1,3,3) = 1+1 = 2$

$\therefore M_{\bar{S}}(0,3,3) = M_{\bar{S}}(1,3,3) = M_{\bar{S}}(2,3,3) = 2 = \frac{1}{3}.6 = \frac{1}{3}\overline{spt}(3)$. Hence The Result.

Now from above table we get; $\sum \omega(\vec{\pi}) = 6$

$$i.e., \sum_{k=0}^{2} M_{\bar{S}}(k,3,3) = 6$$

$$\therefore \overline{spt}(3) = \sum_{k=0}^{2} M_{\bar{S}}(k,3,3) = \sum \omega(\vec{\pi}).$$

We define $M_{\bar{S}}(k,t,m) = \sum_{m \equiv k (\mathrm{mod}\, t)} M_{\bar{S}}(m,n)$

and $\overline{spt}(n) = \sum_{m=-\infty}^{\infty} M_{\bar{S}}(m,n) = \sum_{k=0}^{t-1} M_{\bar{s}}(k,t,n).$

Result 2: $N_{\bar{S}_1}(0,3,3) = N_{\bar{S}_1}(1,3,3) = N_{\bar{S}_1}(2,3,3) = \frac{1}{3}\overline{spt_1}(3)$

Proof: We prove the result with an example. We see the vector partitions from $\overline{S_1}$ of 3 along with their weights and cranks are given as follows:

Table 5:

$\overline{S_1}$ -vector partition $(\vec{\pi})$ of 3	Weight ω $(\vec{\pi})$	Crank c $(\vec{\pi})$	(mod 3)
$\vec{\pi}_1 = (3,\phi,\phi,\phi)$	+ 1	0	0
$\vec{\pi}_2 = (1+2,\phi,\phi,\phi)$	–1	0	0
$\vec{\pi}_3 = (1,2,\phi,\phi)$	+1	1	1
$\vec{\pi}_4 = (1,\phi,2,\phi)$	+1	–1	2
$\vec{\pi}_5 = (1,1,1,\phi)$	+1	0	0
$\vec{\pi}_6 = (1,1+1,\phi,\phi)$	+1	2	2
$\vec{\pi}_7 = (1,\phi,1+1,\phi)$	+1	–2	1
$\vec{\pi}_8 = (1,\phi,\phi,2)$	+1	0	0
	$\sum \omega(\vec{\pi}) = 6$		

Here we have used ϕ to indicate the empty partition.

Thus we have, $N_{\overline{S_1}}(0,3,3) = +1+1-1+1 = 2$, $N_{\overline{S_1}}(1,3,3) = 1+1 = 2$, $N_{\overline{S_1}}(2,3,3) = 1+1 = 2$

$\therefore N_{\overline{S_1}}(0,3,3) = N_{\overline{S_1}}(1,3,3) = N_{\overline{S_1}}(2,3,3) = 2 = \frac{1}{3}.6 = \frac{1}{3}\overline{spt_1}(3)$. Hence The Result.

Now from above table we get; $\sum \omega(\vec{\pi}) = 6$

$$i.e. \sum_{k=0}^{2} N_{\overline{S_1}}(k,3,3) = 6$$

$$\therefore \overline{spt_1}(3) = \sum_{k=0}^{2} N_{\overline{S_1}}(k,3,3) = \sum \omega(\vec{\pi}).$$

Now we can define $N_{\overline{S_1}}(k,t,n) = \sum_{m \equiv k (\text{mod}\, t)} N_{\overline{S_1}}(m,n)$

and $\overline{spt_1}(n) = \sum_{m=-\infty}^{\infty} N_{\overline{S_1}}(m,n) = \sum_{k=0}^{t-1} N_{\overline{S_1}}(k,t,n).$

Result 3: $M_{\overline{S_2}}(0,3,4) = M_{\overline{S_2}}(1,3,4) = M_{\overline{S_2}}(2,3,4) = \frac{1}{3}\overline{spt_2}(4)$.

Proof: We prove the result with the help of examples. We see the vector partitions from $\overline{S_2}$ of 4 along with their weights and cranks and are given as follows:

Table 6:

$\overline{S_2}$-vector partition $(\vec{\pi})$ of 4	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$	mod 3
$\vec{\pi}_1 = (4, \phi, \phi, \phi)$	1	0	0
$\vec{\pi}_2 = (2+2, \phi, \phi)$	1	1	1
$\vec{\pi}_3 = (2, \phi, 2, \phi)$	1	–1	2
	$\sum \omega(\vec{\pi}) = 3$		

Here we have used ϕ to indicate the empty partition. Thus we have,

$$M_{\overline{S_2}}(0,3,4) = 1, \quad M_{\overline{S_2}}(1,3,4) = 1,$$

$$M_{\overline{S_2}}(2,3,4) = M_{\overline{S_2}}(-1,3,4) = 1$$

$$\therefore M_{\overline{S_2}}(0,3,4) = M_{\overline{S_2}}(1,3,4)$$

$$= M_{\overline{S_2}}(2,3,4) = 1 = \frac{1}{3}.3 = \frac{1}{3}\overline{spt_2}(3)$$. Hence The Result.

Now from above table we get; $\sum \omega(\vec{\pi}) = 3$, i.e., $\sum_{k=0}^{2} M_{\overline{S_2}}(k,3,4) = 3$.

$$\therefore \overline{spt_2}(4) = \sum_{k=0}^{2} M_{\overline{S_2}}(k,3,4) = \sum \omega(\vec{\pi}).$$

Now we can define;

$$M_{\overline{S_2}}(k,t,n) = \sum_{m \equiv k \pmod t} M_{\overline{S_2}}(m,n)$$

and $\overline{spt_2}(n) = \sum_{m=-\infty}^{\infty} M_{\overline{S_2}}(m,n) = \sum_{k=0}^{t-1} M_{\overline{S_2}}(k,t,n).$

Theorem 1: The number of vector partitions of n in $\bar{S}$ with crank m counted according to the weight ω is non-negative. i.e. $M_{\bar{S}}(m,n) \ge 0$.

Proof: The generating function for $M_{\bar{S}}(m,n)$ is given by $\sum_{n=1}^{\infty} \sum_{m} M_{\bar{S}}(m,n) z^m x^n$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{n+1};x)_\infty (-x^{n+1};x)_\infty}{(zx^n;x)_\infty (z^{-1}x^n;x)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{(zx^n;x)_\infty (z^{-1}x^n;x)_\infty}.(x^{n+1};x)_\infty (-x^{n+1};x)_\infty$$

$$= \sum_{n=1}^{\infty} \frac{x^n \left(x^{2n+2};x^2\right)_\infty}{\left(zx^n;x\right)_\infty \left(z^{-1}x^n;x\right)_\infty}$$

$$[\text{since} \quad \sum_{n=1}^{\infty} (x^{n+1};x)_{\infty}(-x^{n+1};x)_{\infty}$$

$$= (x^2;x)_{\infty}(-x^2;x)_{\infty} + (x^3;x)_{\infty}(-x^3;x)_{\infty} + (x^4;x)_{\infty}(-x^4;x)_{\infty} + ...$$

$$= (1-x^2)(1-x^3)...(1+x^2)(1+x^3)... + (1-x^3)(1-x^4).....(1+x^3)(1+x^4)....$$

$$= (1-x^4)(1-x^6)... + (1-x^6)(1-x^8) + ... = \sum_{n=1}^{\infty} (x^{2n+2};x^2)_{\infty}]$$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{2n};x)_{\infty}}{(zx^n;x)_{\infty}(z^{-1}x^n;x)_{\infty}} \cdot \frac{(x^{2n+2};x^2)}{(x^{2n};x)_{\infty}}$$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{2n};x)_{\infty}}{(zx^n;x)_{\infty}(z^{-1}x^n;x)_{\infty}} \cdot \frac{1}{(1-x^{2n})(x^{2n+1};x^2)_{\infty}}$$

$$[\text{since} \ \sum_{n=1}^{\infty} \frac{(x^{2n+2};x^2)_{\infty}}{(x^{2n};x)_{\infty}} = \frac{(x^4;x^2)_{\infty}}{(x^2;x)_{\infty}} + \frac{(x^6;x^2)_{\infty}}{(x^4;x)_{\infty}} + ...$$

$$= \frac{(1-x^4)(1-x^6)...}{(1-x^2)(1-x^3)(1-x^4)...} + \frac{(1-x^6)(1-x^8)...}{(1-x^4)(1-x^5)...}$$

$$= \frac{1}{(1-x^2)} \cdot \frac{1}{(1-x^3)(1-x^5)..} + \frac{1}{(1-x^4)} \cdot \frac{1}{(1-x^5)(1-x^7)..} + ...$$

$$= \sum_{n=1}^{\infty} \frac{1}{1-x^{2n}} \cdot \frac{1}{(x^{2n+1};x^2)_{\infty}}]$$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{2n};x)_{\infty}}{(zx^n;x)_{\infty}(z^{-1}x^n;x)_{\infty}} \cdot \frac{1}{(1-x^{2n})(x^{2n+1};x^2)_{\infty}}$$

$$= \sum_{n=1}^{\infty} x^n \sum_{k=0}^{\infty} \frac{(z^{-1}x^n)^k}{(zx^{n+k};x)_{\infty}(x)_k} \cdot \frac{1}{(1-x^{2n})(x^{2n+1};x^2)_{\infty}}$$

$$[\text{since} = \sum_{n=1}^{\infty} \frac{x^n.(x^{2n};x)_{\infty}}{(zx^n;x)_{\infty}(z^{-1}x^n;x)_{\infty}} = \sum_{n=1}^{\infty} x^n \sum_{k=0}^{\infty} \frac{(z^{-1}x^n)^k}{(zx^{n+k};x)_{\infty}(x)_k}] \text{ (by [3]).}$$

We see that the co-efficient of any power x in right hand side is non-negative so the

co-efficient $M_{\bar{S}}(m,n)$ of $z^m x^n$ is non-negative, i.e. $M_{\bar{S}}(m,n) \geq 0$. Hence the Theorem.

Numerical example 1: The vector partitions from $\bar{S}$ of 4 along with their weights and cranks are given as follows:

Table 7:

$\bar{S}$-vector partition $(\vec{\pi})$ of 4	Weight $\omega(\vec{\pi})$	Crank $c(\vec{\pi})$

$\vec{\pi}_1 = (4,\phi,\phi,\phi)$	+ 1	0
$\vec{\pi}_2 = (3+1,\phi,\phi,\phi)$	-1	0
$\vec{\pi}_3 = (1,3,\phi,\phi)$	+1	1
$\vec{\pi}_4 = (1,\phi,3,\phi)$	+1	-1
$\vec{\pi}_5 = (1,\phi,\phi,3)$	+1	0
$\vec{\pi}_6 = (2,2,\phi,\phi)$	+1	1
$\vec{\pi}_7 = (2,\phi,2,\phi)$	+1	-1
$\vec{\pi}_8 = (1+2, 1,\phi,\phi)$	-1	1
$\vec{\pi}_9 = (1+2,\phi,1,\phi)$	-1	-1
$\vec{\pi}_{10} = (1, 1, 2, \phi)$	+1	0
$\vec{\pi}_{11} = (1,2,1,\phi)$	+1	0
$\vec{\pi}_{12} = (1,1,\phi,2)$	+1	1
$\vec{\pi}_{13} = (1,\phi,1,2)$	+1	-1
$\vec{\pi}_{14} = (1,1+2,\phi,\phi)$	+1	2
$\vec{\pi}_{15} = (1,\phi,1+2,\phi)$	+1	-2
$\vec{\pi}_{16} = (1,1+1+1,\phi,\phi)$	+1	3
$\vec{\pi}_{17} = (1,\phi,1+1+1,\phi)$	+1	-3
$\vec{\pi}_{18} = (1,1+1,1,\phi)$	+1	1
$\vec{\pi}_{19} = (1,1,1+1,\phi)$	+1	-1

	$\sum \omega(\vec{\pi}) = 13$	

Here we have used ϕ to indicate the empty partition. Thus we have;

$M_{\bar{S}}(0,4) = 3,\ M_{\bar{S}}(1,4) = 3,\ M_{\bar{S}}(-1,4) = 3,\ M_{\bar{S}}(2,4) = 1,\ M_{\bar{S}}(-2,4) = 1,\ M_{\bar{S}}(3,4) = 1,$ and $M_{\bar{S}}(-3,4) = 1,$

$\therefore \sum_m M_{\bar{S}}(m,4) = 13,$ i.e. every term is non-negative.

$\therefore\ M_{\bar{S}}(m,4)) \geq 0.$ But we have already found that $\sum_m M_{\bar{S}}(m,3) = 6,$

i.e., every term is non-negative. $\therefore\ M_{\bar{S}}(m,3) \geq 0.$

So we can conclude that; $M_{\bar{S}}(m,n) \geq 0.$

Theorem 2: The number of vector partitions of n in $\overline{S_1}$ with crank m counted according to the weight ω is non-negative. i.e. $N_{\bar{S}_1}(m,n) \geq 0.$

Proof: The generating function for $N_{\bar{S}_1}(m,n)$ is given by

$$\sum_{n=1}^{\infty} \sum_m N_{\bar{S}_1}(m,n) z^m x^m = \sum_{n=0}^{\infty} \frac{x^{2n+1}(x^{2n+2};x)_\infty(-x^{2n+2};x)_\infty}{(zx^{2n+1};x)_\infty(z^{-1}x^{2n+1};x)_\infty}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(zx^{2n+1};x)_\infty(z^{-1}x^{2n+1};x)_\infty} .(x^{4n+4};x^2)_\infty$$

[since $\sum_{n=0}^{\infty} (x^{2n+2};x)_\infty(-x^{2n+2};x)_\infty$

$$= (x^2;x)_\infty(-x^2;x)_\infty + (x^4;x)_\infty(-x^4;x)_\infty + ...$$
$$= (1-x^2)(1-x^3)....(1+x^2)(1+x^3)... + (1-x^4)(1-x^5)...(1+x^4)(1+x^5)... + ...$$
$$= (1-x^4)(1-x^6).... + (1-x^8)(1-x^{10})... + (1-x^{12})... + ...$$
$$= (x^4;x^2)_\infty + (x^8;x^2)_\infty + (x^{12};x^2)_\infty + ... = \sum_{n=0}^{\infty} (x^{4n+4};x^2)_\infty]$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}(x^{4n+4};x)_\infty}{(zx^{2n+1};x)_\infty(z^{-1}x^{2n+1};x)_\infty} . \frac{(x^{4n+4};x^2)_\infty}{(x^{4n+4};x)_\infty}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}(x^{4n+4};x)_\infty}{(zx^{2n+1};x)_\infty(z^{-1}x^{2n+1};x)_\infty} . \frac{1}{(x^{4n+5};x^2)_\infty}$$

[since $\sum_{n=0}^{\infty} \frac{(x^{4n+4};x^2)_\infty}{(x^{4n+4};x)_\infty} = \frac{(x^4;x^2)_\infty}{(x^4;x)_\infty} + \frac{(x^8;x^2)_\infty}{(x^8;x)_\infty} + ...$

$$=\frac{(1-x^4)(1-x^6)...}{(1-x^4)(1-x^5)(1-x^6)...}+\frac{(1-x^8)(1-x^{10})...}{(1-x^8)(1-x^9).....}+...$$

$$=\frac{1}{(1-x^5)(1-x^7)...}+\frac{1}{(1-x^9)(1-x^{11})...}+...$$

$$=\sum_{n=0}^{\infty}\frac{1}{(x^{4n+5};x^2)_\infty}]$$

$$=\sum_{n=0}^{\infty}x^{2n+1}\sum_{k=0}^{\infty}\frac{(z^{-1}x^{2n+1})^k}{(zx^{2n+1+k};x)_\infty(x)_k}\cdot\frac{1}{(x^{4n+5};x^2)_\infty}$$

[since $\sum_{n=0}^{\infty}x^{2n+1}\frac{(x^{4n+4};x)_\infty}{(zx^{4n+4};x)_\infty(z^{-1}x^{4n+4};x)_\infty}=\sum_{n=0}^{\infty}x^{2n+1}\sum_{k=0}^{\infty}\frac{(z^{-1}x^{2n+1})^k}{(zx^{2n+1+k};x)_\infty(x)_k}$]. (by [3])

We see that the co-efficient of any power x in the right hand side is non-negative so the co-efficient $N_{\overline{S_1}}(m,n)$ of $z^m x^n$ is non-negative. i.e. $N_{\overline{S_1}}(m,n)\geq 0$. Hence the Theorem.

Numerical Example 2:

The vector partitions from $\overline{S_1}$ of 3 along with their weights and cranks are given as follows:

Table 8:

$\overline{S_1}$-vector partition $(\vec{\pi})$ of 3	Weight $\omega(\vec{\pi})$	Crank $c(\vec{\pi})$
$\vec{\pi}_1=(1,\phi,\phi,2)$	+ 1	0
$\vec{\pi}_2=(1, 1, 1, \phi)$	+1	0
$\vec{\pi}_3=(1+2,\phi,\phi)$	–1	0
$\vec{\pi}_4=(3,\phi,\phi,\phi)$	+1	0
$\vec{\pi}_5=(1,\phi,1+1,\phi)$	+1	–2
$\vec{\pi}_6=(1,2,\phi,\phi)$	+1	1
$\vec{\pi}_7=(1,\phi,2,\phi)$	+1	–1
$\vec{\pi}_8=(1, 1+1,\phi,\phi)$	+1	2

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	$\sum \omega = 6$	

Here we have used ϕ to indicate the empty partition. Thus we have

$N_{\bar{S_1}}(0,3)=2,\ N_{\bar{S_1}}(-1,3)=1,\ N_{\bar{S_1}}(1,3)=1,\ N_{\bar{S_1}}(-2,3)=1,\ N_{\bar{S_1}}(2,3)=1,$

$\therefore$ every term is non-negative and $N_{\bar{S_1}}(m,3)=6,$ i.e. $N_{\bar{S_1}}(m,n)\geq 0.$

We can conclude that $N_{\bar{S_1}}(m,n)\geq 0.$

Theorem 3: The number of vector partitions of *n* in $\overline{S_2}$ with crank *m* counted according to the weight ω is non-negative, i.e., $M_{\bar{S}_2}(m,n)\geq 0.$

Proof: The generating function for $M_{\bar{S}_2}(m,n)$ is given by; $\sum_{n=1}^{\infty}\sum_{m=-\infty} M_{\overline{S_2}}(m,n)z^m x^n$

$$=\sum_{n=1}^{\infty}\frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}=\sum_{n=1}^{\infty}\frac{x^{2n}}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.(x^{4n+2};x^2)_\infty.$$

[Since $\sum_{n=1}^{\infty}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty=(x^3;x)_\infty(-x^3;x)_\infty+(x^5;x)_\infty(-x^5;x)_\infty+...$

$$=(1-x^3)(1-x^4)...(1+x^3)(1+x^4)...+(1-x^5)(1-x^6)...(1+x^5)...+...$$
$$=(1-x^6)(1-x^8)...+(1-x^{10})(1-x^{12})...+(1-x^{14})...+...$$

$$=(x^6;x^2)_\infty+(x^{10};x^2)_\infty+...=\sum_{n=1}^{\infty}(x^{4n+2};x^2)_\infty]$$

$$=\sum_{n=1}^{\infty}\frac{x^{2n}(x^{4n};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.\frac{(x^{4n+2};x^2)_\infty}{(x^{4n};x)_\infty}=\sum_{n=1}^{\infty}\frac{x^{2n}(x^{4n};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.\frac{1}{(1-x^{4n})(x^{4n+1};x^2)_\infty}$$

[Since, $\sum_{n=1}^{\infty}\frac{(x^{4n+2};x^2)_\infty}{(x^{4n};x)_\infty}=\frac{(x^6;x^2)_\infty}{(x^4;x)_\infty}+\frac{(x^{10};x^2)_\infty}{(x^8;x)_\infty}+...=\frac{(1-x^6)(1-x^8)...}{(1-x^4)(1-x^5)(1-x^6)...}+$

$$\frac{(1-x^{10})(1-x^{12})...}{(1-x^8)(1-x^9)(1-x^{10})(1-x^{11})...}+...$$

$$=\frac{1}{(1-x^4)(1-x^5)(1-x^7)...}+\frac{1}{(1-x^8)(1-x^9)(1-x^{11})...}+...$$

$$=\sum_{n=1}^{\infty}\frac{1}{1-x^{4n}}.\frac{1}{(x^{4n+1};x^2)_\infty}]$$

$$=\sum_{n=1}^{\infty}x^{2n}\sum_{k=0}^{\infty}\frac{(z^{-1}x^{2n})^k}{(zx^{2n+k};x)_\infty(x)_k}.\frac{1}{(1-x^{4n})(x^{4n+1};x^2)_\infty}$$

[Since, $\sum_{n=1}^{\infty}\frac{x^{2n}(x^{4n};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}=\sum_{n=1}^{\infty}x^{2n}\sum_{k=0}^{\infty}\frac{(z^{-1}x^{2n})^k}{(zx^{2n+k};x)_\infty(x)_k}$]. (by [3])

We see that the coefficient of any power x in the right hand side is non-negative so the coefficient $M_{\bar{S}_2}(m,n)$ of $z^m x^n$ is non-negative, i.e., $M_{\overline{S_2}}(m,n) \geq 0.$ Hence the Theorem.

Numerical Example 3:

The vector partitions from $\overline{S_2}$ of 5 along with their weights and cranks are given as follows:

Table 9:

$\overline{S_2}$-vector partition $(\vec{\pi})$ of 5	Weight $\omega\ (\vec{\pi})$	Crank c$(\vec{\pi})$
$\vec{\pi}_1 = (3+2, \phi, \phi, \phi)$	–1	0
$\vec{\pi}_2 = (2, \phi, \phi, 3)$	1	0
$\vec{\pi}_3 = (2, 3, \phi, \phi)$	1	1
$\vec{\pi}_4 = (2, \phi, 3, \phi)$	1	–1
	$\sum \omega(\vec{\pi}) = 2$	

Here we have used ϕ to indicate the empty partition. Thus we have;

$M_{\overline{S_2}}(0,5) = 1 - 1 = 0,\ M_{\overline{S_2}}(1,5) = 1,$ and $M_{\overline{S_2}}(-1,5) = 1,$ i.e., $\sum_m M_{\overline{S_2}}(m,5) = 2$,

i.e., every term is non-negative, i.e., $M_{\overline{S_2}}(m,n) \geq 0.$

So we can conclude that, $M_{\bar{S}_2}(m,n) \geq 0.$

Corollary 4: $\bar{S}(1,x) = \sum_{n=1}^{\infty} \overline{spt}\ (n)x^n$.

Proof: We get $\bar{S}\ (z,x) = \sum_{n=1}^{\infty} \frac{x^n(-x^{n+1};x)_\infty (x^{n+1};x)_\infty}{(zx^n;x)_\infty (z^{-1}x^n;x)_\infty}$ ([2]).

If z = 1, then we get; $\bar{S}(1,x) = \sum_{n=1}^{\infty} \frac{x^n(-x^{n+1};x)_\infty (x^{n+1};x)_\infty}{(x^n;x)_\infty (x^n;x)_\infty}$

$$= \frac{x(-x^2;x)_\infty (x^2;x)_\infty}{(x;x)^2{}_\infty} + \frac{x^2(-x^3;x)_\infty (x^3;x)_\infty}{(x^2;x)^2{}_\infty} + \ldots$$

$$= \frac{x(-x^2;x)_\infty (1-x^2)(1-x^3)(1-x^4)\ldots}{(1-x)^2(1-x^2)^2(1-x^3)^2\ldots} + \frac{x^2(-x^3;x)_\infty (1-x^3)(1-x^4)\ldots}{(1-x^2)^2(1-x^3)^2\left(1-x^4\right)^2\ldots} + \ldots$$

$$= \frac{x(-x^2;x)_\infty}{(1-x)^2(1-x^2)(1-x^3)\ldots} + \frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(1-x^3)(1-x^4)\ldots} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{x^n(-x^{n+1};x)_\infty}{(1-x^n)^2(x^{n+1};x)_\infty} = \sum_{\text{n}=1}^{\infty} \overline{spt}\ (n)x^n ,$$

i.e; $\overline{S}(1,x)=\sum_{n=1}^{\infty}\overline{spt}(n)x^n$. Hence The Corollary.

Corollary 5: $\overline{S}_1(1,x)=\sum_{n=1}^{\infty}\overline{spt_1}(n)x^2$

Proof: we get $\overline{S}_1(z,x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}(-x^{2n+2};x)_\infty(x^{2n+2};x)_\infty}{(zx^{2n+1};x)_\infty(z^{-1}x^{2n+1};x)_\infty}$ ([2]).

If $z=1$, then we get; $\overline{S}_1(1,x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}(-x^{2n+2};x)_\infty(x^{2n+2};x)_\infty}{(x^{2n+1};x)_\infty(x^{2n+1};x)_\infty}$

$$=\frac{x(-x^2;x)_\infty(x^2;x)_\infty}{(x;x)_\infty(x;x)_\infty}+\frac{x^3(-x^4;x)_\infty(x^4;x)_\infty}{(x^3;x)_\infty(x^3;x)_\infty}+\ldots$$

$$=\frac{x(-x^2;x)_\infty(1-x^2)(1-x^3)(1-x^4)\ldots}{(1-x)^2(1-x^2)^2(1-x^3)^2\ldots}+\frac{x^3(-x^4;x)_\infty(1-x^4)(1-x^5)\ldots}{(1-x^3)^2(1-x^4)^2\ldots}+\ldots$$

$$=\frac{x(-x^2;x)_\infty}{(1-x)^2(1-x^2)(1-x^3)\ldots}+\frac{x^3(-x^4;x)_\infty}{(1-x^3)^2(1-x^4)(1-x^5)\ldots}+\ldots$$

$$=\sum_{n=0}^{\infty}\frac{x^{2n+1}(-x^{2n+2};x)_\infty}{(1-x^{2n+1})^2(x^{2n+2};x)_\infty}$$

i.e; $\overline{S}_1(1,x)=\sum_{n=1}^{\infty}\overline{spt_1}(n)x^n$. Hence The Corollary.

Corollary 6: $\overline{S}_2(1,x)=\sum_{n=1}^{\infty}\overline{spt_2}(n)x^n$.

Proof: We get; $\overline{S}_2(z,x)=\sum_{n=1}^{\infty}\frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}$ [2].

If $z=1$, then we get; $\overline{S}_2(1,x)=\sum_{n=1}^{\infty}\frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(x^{2n};x)_\infty(x^{2n};x)_\infty}$

$$=\frac{x^2(x^3;x)_\infty(-x^3;x)_\infty}{(x^2;x)_\infty^2}+\frac{x^4(-x^5;x)_\infty(x^5;x)_\infty}{(x^4;x)_\infty^2}+\ldots$$

$$=\frac{x^2(-x^3;x)_\infty(1-x^3)(1-x^4)\ldots}{(1-x^2)^2(1-x^3)^2\ldots}+\frac{x^4(-x^5;x)_\infty(1-x^5)(1-x^6)\ldots}{(1-x^4)^2(1-x^5)^2\ldots}+\ldots$$

$$=\frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(1-x^3)\ldots}+\frac{x^4(-x^5;x)_\infty}{(1-x^4)^2(1-x^5)\ldots}+\ldots$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \overline{spt_2}(n)\, x^n .$$

i.e., $\overline{S}_2(1,x) = \sum_{n=1}^{\infty} \overline{spt_2}(n)\, x^n$. Hence The Corollary.

Theorem 4: $\overline{spt}(n) = \sum_{\substack{\bar{\lambda}\in \overline{SP} \\ |\bar{\lambda}|=|\lambda_1|+|\lambda_2|=n}} 1$.

Proof: First we define the $\overline{\text{sptcrank}}$ in terms of partition pairs.

$\overline{SP} = \{\bar{\lambda} = (\lambda_1, \lambda_2) \in P \times P : 1 < s(\lambda_1) \le s(\lambda_2)$ and all parts of λ_2 that are $\ge 2s(\lambda_1)+1$ are odd}.

The generating function for $\overline{spt}(n)$ is given by

$$\sum_{n=1}^{\infty}\overline{spt}(n)x^n = \sum_{n=1}^{\infty} \frac{x^n(-x^{n+1};x)_\infty}{(1-x^n)^2(x^{n+1};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)^2(x^{n+1};x)_\infty}(-x^{n+1};x)_\infty$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)^2(x^{n+1};x)_\infty}.\frac{(x^{2n+2};x^2)_\infty}{(x^{n+1};x)_\infty}$$

[since $\sum_{n=1}^{\infty}(-x^{n+1};x)_\infty = (-x^2;x)_\infty + (-x^3;x)_\infty + \ldots$

$$= (1+x^2)(1+x^3)\ldots.. + (1+x^3)(1+x^4)\ldots. + (1+x^4)\ldots$$

$$= \frac{(1-x^4)(1-x^6)\ldots}{(1-x^2)(1-x^3)(1-x^4)\ldots} + \frac{(1-x^6)(1-x^8)\ldots}{(1-x^3)(1-x^4)\ldots} + \ldots..$$

$$= \frac{(x^4;x^2)_\infty}{(x^2;x)_\infty} + \frac{(x^6;x^2)_\infty}{(x^3;x)_\infty} + \ldots = \sum_{n=1}^{\infty} \frac{(x^{2n+2};x^2)_\infty}{(x^{n+1};x)_\infty}]$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{(x^n;x)_\infty(1-x^n)}.\frac{(x^{2n+2};x^2)_\infty}{(x^{n+1};x)_\infty}$$

[since $\sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)^2(x^{n+1};x)_\infty} = \frac{x}{(1-x)^2(x^2;x)_\infty} + \frac{x^2}{(1-x^2)^2(x^3;x)_\infty} + ..$

$$= \frac{x}{(1-x)^2(1-x^2)(1-x^3)\ldots} + \frac{x^2}{(1-x^2)(1-x^3)(1-x^4)\ldots} + \ldots$$

$$= \frac{x}{(1-x)(x;x)_\infty} + \frac{x^2}{(1-x^2)(x^2;x)_\infty} + \ldots = \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)(x^n;x)_\infty}]$$

$$=\sum_{n=1}^{\infty}\frac{x^n}{(x^n;x)_\infty}\cdot\frac{1}{(1-x^n)}\cdot\frac{(x^{2n+2};x^2)_\infty}{(x^{n+1};x)_\infty}$$

$$=\sum_{n=1}^{\infty}\frac{x^n}{(x^n;x)_\infty}\cdot\frac{1}{(1-x^n)}\cdot\frac{1}{(1-x^{n+1})\dots(1-x^{2n})(x^{2n+1};x^2)_\infty}$$

[since $\sum_{n=1}^{\infty}\frac{(x^{2n+2};x^2)_\infty}{(x^{n+1};x)_\infty}=\frac{(x^4;x^2)_\infty}{(x^2;x)_\infty}+\frac{(x^6;x^2)_\infty}{(x^3;x)_\infty}+\dots$

$$=\frac{(1-x^4)(1-x^6)\dots}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)\dots}+\frac{(1-x^6)(1-x^4)\dots}{(1-x^3)(1-x^4)(1-x^5)\dots}+\dots$$

$$=\frac{1}{(1-x^2)(1-x^3)(1-x^5)(1-x^7)\dots}+\frac{1}{(1-x^3)(1-x^4)(1-x^5)\dots}+\dots$$

$$=\sum_{n=1}^{\infty}\frac{1}{(1-x^{n+1})\dots(1-x^{2n})(x^{2n+1};x^2)_\infty}\;]$$

$$=\sum_{n=1}^{\infty}\frac{x^n}{(x^n;x)_\infty}\cdot\frac{1}{(1-x^n)(1-x^{n+1})\dots(1-x^{2n})(x^{2n+1};x^2)_\infty}$$

$$=\sum_{n=1}^{\infty}\sum_{\substack{\lambda_1\in P\\ s(\lambda_1)=n}}x^{|\lambda_1|}\cdot\sum_{\substack{\lambda_2\in P\\ s(\lambda_2)\ge n}}x^{|\lambda_2|}$$

all parts in $\lambda_2\ge 2n+1$ are odd

$$=\sum_{n=1}^{\infty}\sum_{\substack{\bar{\lambda}\in\overline{SP}\\ |\bar{\lambda}|=|\lambda_1|+|\lambda_2|=n}}x^{|\bar{\lambda}|}\quad .$$

Equating the co-efficient of x^n from both sides we get;

$$\overline{spt}(n)=\sum_{\substack{\bar{\lambda}\in\overline{SP}\\ |\bar{\lambda}|=|\lambda_1|+|\lambda_2|=n}}1$$. Hence The Theorem.

Numerical example 4:

The overpartitions of 3 with smallest parts not overlined are 3, 2+1, $\bar{2}+1$, 1+1+1

Consequently, the number of smallest parts in the overpartitions of 3 with smallest part not overlined is given by $\dot{3}$, $2+\dot{1}$, $\bar{2}+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}$ so that $\overline{spt}(3)=6$

i.e. there are 6 $\overline{SP}$ -partition pairs of 3 like $(3,\phi)$, $(2+1,\phi)$, $(1+1+1,\phi)$, $(1+1,1)$, $(1,1+1)$ and (1,2).

Theorem 5: $\overline{spt_1}(n) = \sum_{\substack{\bar{\lambda}\in \overline{SP_1} \\ |\bar{\lambda}|=|\lambda_1|+|\lambda_2|=n}} 1$

Proof: First we define the $\overline{sptcrank}$ in terms of partition pairs.

$\overline{SP} = \{\bar{\lambda} = (\lambda_1, \lambda_2) \in P \times P : 0 < s(\lambda_1) \le s(\lambda_2)$ and all parts of λ_2 that are $\ge 2s(\lambda_1)+1$ are odd}.

The generating function for $\overline{\mathrm{spt}_1}(\mathrm{n})$ is given by

$$\sum_{n=1}^{\infty} \overline{spt_1}(n)x^n = \sum_{n=0}^{\infty} \frac{x^{2n+1}(-x^{2n+2};x)_\infty}{(1-x^{2n+1})^2(x^{2n+2};x)_\infty} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(1-x^{2n+1})^2(x^{2n+2};x)_\infty}.\sum_{n=1}^{\infty} \frac{(x^{4n};x^2)_\infty}{(x^{2n};x)_\infty}$$

[since $\sum_{n=0}^{\infty}(-x^{2n+2};x)_\infty = (-x^2;x)_\infty + (-x^4;x)_\infty + (-x^6;x)_\infty + ..$

$$= (1+x^2)(1+x^3)(1+x^4).....+(1+x^4)(1+x^5)....+(1+x^6)...+...$$

$$= \frac{(1-x^4)(1-x^6)..}{(1-x^2)(1-x^3)...} + \frac{(1-x^8)(1-x^{10})...}{(1-x^4)(1-x^5)..} + \frac{(1-x^{12})..}{(1-x^6)..}...+..$$

$$= \frac{(x^4;x^2)_\infty}{(x^2;x)_\infty} + \frac{(x^8;x^2)_\infty}{(x^4;x)_\infty} + \frac{(x^{12};x^2)_\infty}{(x^6;x)_\infty} + .. = \sum_{n=1}^{\infty} \frac{(x^{4n};x^2)_\infty}{(x^{2n};x)_\infty}]$$

$$= \frac{x}{(1-x)(1-x^2)(1-x^3)...(1-x)(1-x^2)(1-x^3)(1-x^5)...}$$

$$+ \frac{x^2}{(1-x^3)(1-x^4)...(1-x^3)(1-x^4)...(1-x^7)(1-x^9)...}$$

$$+ \frac{x^5}{(1-x^5)(1-x^6)...(1-x^5)(1-x^6)...(1-x^{11})(1-x^{13})...} + ...$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(x^{2n-1};x)_\infty}.\frac{1}{(1-x^{2n-1})(1-x^{2n})...(1-x^{4n-2})(x^{4n-1};x^2)_\infty}$$

$$= \sum_{n=1}^{\infty} \sum_{\lambda_1 \in P} x^{|\lambda_1|} \sum_{\lambda_2 \in P} x^{|\lambda_2|}$$

$$s(\lambda_1) = n \qquad s(\lambda_2) \ge n$$

all parts in $\lambda_2 \ge 2n+1$ are odd

$$= \sum_{n=1}^{\infty} \sum_{\substack{\bar{\lambda}\in \overline{SP_1} \\ |\bar{\lambda}|=|\lambda_1|+|\lambda_2|=n}} x^{|\bar{\lambda}|} \quad .$$

Equating the co-efficient of x^n from both sides we get;

$$\overline{spt_1}(n) = \sum_{\substack{\bar{\lambda}\in \overline{SP_1} \\ |\bar{\lambda}|=|\lambda_1|+|\lambda_2|=n}} 1$$. Hence The Theorem.

Numerical Example 5:

The overpartitions of 4 with smallest part not overlined and odd are 3+1, $\bar{3}+1$, 2+1+1, $\bar{2}+1+1$ and 1+1+1+1.

Consequently, the number of smallest parts in the overpartitions of 4 with smallest part not overlined and odd is given by; $3+\dot{1}$, $\bar{3}+\dot{1}$, $2+\dot{1}+\dot{1}$, $\bar{2}+\dot{1}+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}+\dot{1}$ so that $\overline{spt_1}(4)=10$ i.e., there are 10 $\overline{SP_1}$ -partition pairs of 4 like-

$(3+1,\phi)$, $(1,3)$, $(2+1+1,\phi)$, $(2+1,1)$, $(1,1+2)$, $(1+1,2)$, $(1+1+1+1,\phi)$, (1+1+1,1),(1+1,1+1) and (1, 1+1+1).

Theorem 6: $\overline{spt_2}(n) = \sum_{\substack{\bar{\lambda}\in \overline{SP_2} \\ |\bar{\lambda}|=|\lambda_1|+|\lambda_2|=n}} 1$

Proof: First we define the $\overline{sptcrank}$ in terms of partition pairs,

$\overline{SP} = \{\bar{\lambda} = (\lambda_1, \lambda_2) \in P \times P : 0 < s(\lambda_1) \le s(\lambda_2)$ and all parts of λ_2 that are $\ge 2s(\lambda_1)+1$ are odd}.

Let $\overline{SP}_2$ be the set of $\vec{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}$ with $s(\lambda_1)$ even. The generating function for $\overline{spt}_2(n)$ is given by; $\sum_{n=1}^{\infty} \overline{spt}_2(n)x^n = \sum_{n=1}^{\infty} \dfrac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$

$$=\sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty}(-x^{2n+1};x)_\infty$$

$$=\sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty}.\frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}$$

[Since, $\sum_{n=1}^{\infty}(-x^{2n+1};x)_\infty = (-x^3;x)_\infty + (-x^5;x)_\infty + \ldots$

$$=(1+x^3)(1+x^4)\ldots+(1+x^5)(1+x^6)\ldots+(1+x^7)(1+x^8)\ldots+\ldots$$

$$=\frac{(1-x^6)(1-x^8)\ldots}{(1-x^3)(1-x^4)\ldots}+\frac{(1-x^{10})(1-x^{12})\ldots}{(1-x^5)(1-x^6)\ldots}+\frac{(1-x^{14})\ldots}{(1-x^7)\ldots}+\ldots$$

$$=\frac{(x^6;x^2)_\infty}{(x^3;x)_\infty}+\frac{(x^{10};x^2)_\infty}{(x^5;x)_\infty}+\ldots=\sum_{n=1}^{\infty}\frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}]$$

$$=\sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty}.\frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty} = \sum_{n=1}^{\infty}\frac{x^{2n}}{(x^{2n};x)_\infty}.\frac{1}{(1-x^{2n})}.\frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}$$

[Since, $\sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty} = \frac{x^2}{(1-x^2)^2(x^3;x)_\infty} + \frac{x^4}{(1-x^4)^2(x^5;x)_\infty} + \ldots$

$$= \frac{x^2}{(1-x^2)^2(1-x^3)(1-x^4)\ldots} + \frac{x^4}{(1-x^4)^2(1-x^5)(1-x^6)\ldots} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{(x^{2n};x)_\infty} \cdot \frac{1}{(1-x^{2n})}]$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{(x^{2n};x)_\infty} \cdot \frac{1}{(1-x^{2n})(1-x^{2n+1})\ldots(1-x^{4n})(x^{4n+1};x^2)_\infty}$$

[Since, $\sum_{n=1}^{\infty} \frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty} = \frac{(x^6;x^2)_\infty}{(x^3;x)_\infty} + \frac{(x^{10};x^2)_\infty}{(x^5;x)_\infty} + \ldots$

$$= \frac{(1-x^6)(1-x^8)\ldots}{(1-x^3)(1-x^4)\ldots} + \frac{(1-x^{10})(1-x^{12})\ldots}{(1-x^5)(1-x^6)\ldots(1-x^{10})(1-x^{11})\ldots} + \ldots$$

$$= \frac{1}{(1-x^3)(1-x^4)(1-x^5)\ldots} + \frac{1}{(1-x^5)(1-x^6)\ldots(1-x^9)(1-x^{11})\ldots} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{(1-x^{2n+1})\ldots.(1-x^{4n})(x^{4n+1};x^2)_\infty}]$$

$$= \sum_{n=1}^{\infty} \sum_{\substack{\lambda_1 \in P \\ S(\lambda_1)=n}} x^{|\lambda_1|} \sum_{\substack{\lambda_2 \in P \\ s(\lambda_2)\geq n}} x^{|\lambda_2|}$$

all parts in $\lambda_2 \geq 2n+1$ are odd

$$= \sum_{n=1}^{\infty} \sum_{\substack{\bar{\lambda} \in \overline{SP}_2 \\ |\bar{\lambda}|=|\lambda_1|+|\lambda_2|=n}} x^{|\bar{\lambda}|}.$$

Equating the co-efficient of x^n from both sides we get;

$\overline{spt_2}(n) = \sum_{\substack{\bar{\lambda} \in \overline{SP}_2 \\ |\bar{\lambda}|=|\lambda_1|+|\lambda_2|=n}} 1$. Hence The Theorem.

Numerical Example 6:

The overpartitions of 6 with smallest parts not overlined and even are 6, 4+2, $\bar{4}+2$, and 2+2+2. Consequently, the number of smallest parts in the overpartitions of 6 with smallest part not overlined and even is given by;

$\dot{6}$ $4+\dot{2}$, $\bar{4}+\dot{2}$, $\dot{2}+\dot{2}+\dot{2}$, so that $\overline{spt}_2(6)=6$ i.e., there are $6\ \overline{SP}_2$ -partition pairs of 6 like:

$(6,\phi)$, $(4+2,\phi)$, $(2,4)$, $(2+2+2,\phi)$, $(2+2,2)$ and (2, 2+2).

Result 4: $M_{\bar{S}}(0,3,3) = M_{\bar{S}}(1,3,3) = M_{\bar{S}}(2,3,3) = \frac{\overline{\text{spt}}(3)}{3}$.

Proof: First we define a $\overline{\text{crank}}$ of partitions pairs $\vec{\lambda}=(\lambda_1,\lambda_2)\in\overline{SP}$.

For $\vec{\lambda}=(\lambda_1,\lambda_2)\in\overline{SP}$ we define;

$k(\vec{\lambda})$ =# of pairs j in λ_2 such that $s(\lambda_1)\le j\le 2s(\lambda_1)-1, s(\lambda_1)\le s(\lambda_2)$,
and also define

$$\overline{crank}(\vec{\lambda})=\begin{cases}(\#\text{of parts of } \lambda_1\ge s(\lambda_1)+k)-k;\text{if } k>0\\(\#\text{of parts of } \lambda_1)-1;\text{if } k=0\end{cases}$$

where $k=k(\vec{\lambda})$.

Table 10:

$\overline{\text{SP}}$ -partition pair $\vec{\lambda}=(\lambda_1,\lambda_2)$	K	$\overline{\text{crank}}$	(mod3)
$(3,\phi)$	0	0	0
$(2+1,\phi)$	0	1	1
$(1+1+1,\phi)$	0	2	2
$(1+1,1)$	1	-1	2
$(1,1+1)$	2	-2	1
$(1,2)$	0	0	0

From the table we get;

$M_{\bar{S}}(0,3,3) = M_{\bar{S}}(1,3,3) = M_{\bar{S}}(2,3,3) = 2 = \frac{1}{3}6 = \frac{1}{3}\overline{\text{spt}}(3)$. Hence The Result.

Result 5: $N_{\bar{S}_1}(0,5,5) = N_{\bar{S}_1}(1,5,5) = N_{\bar{S}_1}(2,5,5,) = N_{\bar{S}_1}(3,5,5) = N_{\bar{S}_1}(4,5,5) = 4 = \frac{1}{5}\ \overline{spt}_1(5)$

Proof: We prove the result with the help of an example.

We can define a $\overline{crank}$ of partition pairs $\vec{\lambda}=(\lambda_1,\lambda_2)\in\overline{SP}_1$. For $\vec{\lambda}=(\lambda_1,\lambda_2)\in\overline{SP}_1$ we define

$k(\vec{\lambda})$ =# of parts j in λ_2 such that $s(\lambda_1)\le j\le 2\ s(\lambda_1)-1,\ s(\lambda_1)\le s(\lambda_2)$
and also define;

$$\overline{crank}(\vec{\lambda})=\begin{cases}(\#\text{of parts of } \lambda_1\ge s(\lambda_1)+k)-k;\text{if } k>0\\(\#\text{of parts of } \lambda_1)-1;\text{if } k=0\end{cases}$$

where $k = k(\vec{\lambda})$.

The number of smallest parts in the overpartitions of 5 with smallest part not overlined and odd is given by;

$\dot{5}$, $4+\dot{1}$, $\bar{4}+\dot{1}$, $3+\dot{1}+\dot{1}$, $\bar{3}+\dot{1}+\dot{1}$, $2+2+\dot{1}$, $\bar{2}+2+\dot{1}$, $2+\dot{1}+\dot{1}+\dot{1}$, $\bar{2}+\dot{1}+\dot{1}+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}+\dot{1}+\dot{1}$,

so that $\overline{spt}_1(5) = 20$. There are 20 $\overline{SP}_1$ - partition pairs of 5.

Table 11:

$\overline{SP}_1$-partition pair of 5	K	$\overline{crank}$	(mod 5)
(1, 2+2)	0	0	0
(2+1+1,1)	1	0	0
(3+1,1)	1	0	0
(5, ϕ)	0	0	0
(1, 1+1+1+1)	4	–4	1
(1+1, 3)	0	1	1
(2+1,2)	0	1	1
(4+1, ϕ)	0	1	1
(1+1, 1+1+1)	3	–3	2
(1+1+1, 2)	0	2	2
(2+2+1, ϕ)	0	2	2
(3+1+1, ϕ)	0	2	2
(1, 2+1+1)	2	–2	3
(1+1+1, 1+1)	2	–2	3
(2+1, 1+1)	2	–2	3
(2+1+1+1, ϕ)	0	3	3
(1, 3+1)	1	–1	4
(1+1, 2+1)	1	–1	4
(1+1+1+1, 1)	1	–1	4
(1+1+1+1+1, ϕ)	0	4	4

From the table we get; $N_{\bar{S}_1}(0,5,5) = N_{\bar{S}_1}(1,5,5) = N_{\bar{S}_1}(2,5,5) = N_{\bar{S}_1}(3,5,5) =$

$N_{\bar{S}_1}(4,5,5) = 4 = \frac{1}{5}\overline{spt}_1(5)$. Hence The Result.

Result 6: $M_{\overline{S_2}}(0,5,8) = M_{\bar{S}_2}(1,5,8,) = M_{\bar{S}_2}(2,5,8,) = M_{\bar{S}_2}(3,5,8) = M_{\bar{S}_2}(4,5,8) = 3 = \frac{1}{5}\,\overline{spt}_2(8)$.

Proof: We prove the result with the help of examples. We can define a $\overline{crank}$ of partition pairs $\vec{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}_2$.

For $\vec{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP_2}$, we define, $k(\vec{\lambda}) =$ # of pairs *j* in λ_2 such that $s(\lambda_1) \le j \le 2\ s(\lambda_1) - 1$, and also

define; $\overline{crank}(\vec{\lambda}) = \begin{cases} (\#\text{of parts of } \lambda_1 \ge s(\lambda_1) + k) - k; \\ \text{if } k > 0 \\ (\#\text{of parts of } \lambda_1) - 1; \text{if } k = 0 \end{cases}$ where $k = k(\vec{\lambda})$.

We know that $\overline{spt_2}(8) = 15$. There are 15 $\overline{SP_2}$ -partition pairs of 8.

Table 12:

$\overline{SP_2}$ -partition pair of 8	*K*	$\overline{crank}$	(mod 5)
(3+2, 3)	1	0	0
(4+2, 2)	1	0	0
$(8, \phi)$	0	0	0
(2+2, 4)	0	1	1
(4+4, ϕ)	0	1	1
(6+2, ϕ)	0	1	1
(2, 2+2+2)	3	–3	2
(3+3+2, ϕ)	0	2	2
(4+2+2, ϕ)	0	2	2
(2, 3+3)	2	–2	3
(2+2, 2+2)	2	–2	3
(2+2+2+2, ϕ)	0	3	3
(2, 4+2)	1	–1	4
(4, 4)	1	–1	4
(2+2+2, 2)	1	–1	4

From the table we get; $M_{\overline{S}_2}(0,5,8) = M_{\overline{S}_2}(1,5,8,) = M_{\overline{S}_2}(2,5,8,) =$

$$M_{\overline{S}_2}(3,5,8) = M_{\overline{S}_2}(4,5,8) = 3 = \frac{1}{5}\overline{spt}_2(8)$$. Hence The Result.

4. WE WANT TO DESCRIBE THE $\overline{\text{sptcrank}}$ OF A MARKED OVERPARTITION [7]:

To define the $\overline{\text{sptcrank}}$ of a marked overpartition we first need to define a function k (m,n) for positive integers m and n such that $m \geq n+1$ we write $m = b2^j$, where b is odd and $j \geq o$. For a given odd integer b and a positive integer n we define $j_0 = j_0(b,n)$ to be the smallest nonnegative integer j_0 such that $b2^{j_0} \geq n+1$.

$$\text{We define; } k(m,n) = \begin{cases} 0, \text{ if } b \geq 2n \\ 2^{j-j_0} \text{ if } b2^{j_0} < 2n \\ 0, \text{ if } b2^{j_0} = 2n. \end{cases}$$

It is convenient to define k (m,n)=0, if m=0.

If $j \geq 1$ then $b2^{j_0} \leq 2n$ so that the function $k(m,n)$ is well-defined . For a partition

$\pi : m_1 + m_2 + ... + m_t$ into distinct parts and

$m_1 > m_2 > ... > m_t \geq n+1$, we define the function

$$k(\pi,n) = \sum_{j=1}^{t} k(m_j,n) = \sum_{m\in\pi} k(m,n).$$

For a marked overpartitions (π, j) we let π_1 be the partition formed by the non-overlined parts of π, π_2 be the partition (into distinct parts) formed by the overlined parts of π so that $s(\pi_2) > s(\pi_1)$, we define $\overline{k}(\pi,i) = \nu(\pi_1) - j + k(\pi_2, s(\pi_1))$, where $\nu(\pi_1)$ is the number of smallest parts of π_1.

Now we can define;

$$\overline{sptcrank}(\pi, j) = \begin{cases} (\#\text{of parts of } \pi_1 \geq s(\pi_1) + \overline{k}) - \overline{k}, \text{ if } \overline{k} = \overline{k}(\pi, j) > 0 \\ (\#\text{of parts of } \pi_1) - 1; \text{ if } \overline{k} = \overline{k}(\pi, j) = 0. \end{cases}$$

Corollary 7[9]: The residue of the $\overline{\text{sptcrank}}$(mod 3) divides the marked overpartitions of 3n into 3 equal classes.

Proof: We prove the corollary with the help of an example. There are 42 marked overpartitions of 3n (where n = 2) so that $\overline{spt}(6) = 42$.

Table 13:

Marked overpartition (π, j) of 6	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\overline{k}$	$\overline{\text{sptcrank}}$	(mod3)
(6,1)	6	ϕ	1	0	0	0	0

$(\bar{5}+1,1)$	1	5	1	0	0	0	0
$(\bar{4}+2,1)$	2	4	1	0	0	0	0
$(4+1+1,1)$	4+1+1	ϕ	2	0	1	0	0
$(\bar{3}+\bar{2}+1,1)$	1	3+2	1	0	0	0	0
$(3+1+1+1,2)$	3+1+1+1	ϕ	3	0	1	0	0
$(3+1+1+1,3)$	3+1+1+1	ϕ	3	0	0	3	0
$(2+2+1+1,2)$	2+2+1+1	ϕ	2	0	0	3	0
$(\bar{2}+2+1+1,2)$	2+1+1	2	2	0	1	0	0
$(2+1+1+1+1,1)$	2+1+1+1+1	ϕ	4	0	3	-3	0
$(2+1+1+1+1,3)$	2+1+1+1+1	ϕ	4	0	1	0	0
$(\bar{2}+1+1+1+1,1)$	1+1+1+1	2	4	0	3	-3	0
$(\bar{2}+1+1+1+1,4)$	1+1+1+1	2	4	0	0	-3	0
$(1+1+1+1+1+1,3)$	1+1+1+1+1 +1	ϕ	6	0	3	-3	0
$(5+1,,1)$	5+1	ϕ	1	0	0	1	1
$(4+2,1)$	4+2	ϕ	1	0	0	1	1
$(\bar{4}+1+1,2)$	1+1	4	2	0	0	1	1
(3+3,2)	3+3	ϕ	2	0	0	1	1
$(\bar{3}+2+1,1)$	2+1	3	1	0	0	1	1
$(3+\bar{2}+1,1)$	3+1	2	1	0	0	1	1
$(\bar{3}+1+1+1,1)$	1+1+1	3	3	0	2	-2	1
(2+2+2,1)	2+2+2	ϕ	3	0	2	-2	1
(2+2+1+1,1)	2+2+1+1	ϕ	2	0	1	1	1
(2+1+1+1+1,2)	2+1+1+1+1	ϕ	4	0	2	-2	1

(2+1+1+1+1,4)	2+1+1+1+1	ϕ	4	0	0	4	1
$(\bar{2}+1+1+1+1,2)$	1+1+1+1	2	4	0	2	-2	1
(1+1+1+1+1+1,1)	1+1+1+1+1+1	ϕ	6	0	5	-5	1
(1+1+1+1+1+1,4)	1+1+1+1+1+1	ϕ	6	0	2	-2	1
(4+1+1,2)	4+1+1	ϕ	2	0	0	2	2
$(\bar{4}+1+1,1)$	1+1	4	2	0	1	-1	2
$(3+3,1)$	3+3	ϕ	2	0	1	-1	2
(3+2+1,1)	3+2+1	ϕ	1	0	0	2	2
(3+1+1+1,1)	3+1+1+1	ϕ	3	0	2	-1	2
$(\bar{3}+1+1+1,2)$	1+1+1	3	3	0	1	-1	2
$(\bar{3}+1+1+1,3)$	1+1+1	3	3	0	0	2	2
(2+2+2,2)	2+2+2	ϕ	3	0	1	-1	2
(2+2+2,3)	2+2+2	ϕ	3	0	0	2	2
$(\bar{2}+2+1+1,2)$	2+1+1	2	2	0	0	2	2
$(\bar{2}+1+1+1+1,3)$	1+1+1+1	2	4	0	1	-1	2
(1+1+1+1+1+1,2)	(1+1+1+1+1+1)	ϕ	6	0	4	-4	2
(1+1+1+1+1+1,5)	(1+1+1+1+1+1)	ϕ	6	0	1	-1	2
(1+1+1+1+1+1,6)	(1+1+1+1+1+1)	ϕ	6	0	0	5	2

We see that the residue of the $\overline{\mathrm{sptcrank}}$ (mod 3) divides the marked overpartitions of 3*n* into 3 equal classes. Hence the Corollary.

Corollary 8 [9]: The residue of the $\overline{sptcrank}$ (mod 3) divides the marked overpartitions of 3*n* with smallest part not overlined and odd into 3 equal classes.

Proof: We prove the Corollary with the help of example. There are 36 marked overpartitions of 3*n* (when *n* = 2) with the smallest part not overlined and odd so that $\overline{spt_1}(6) = 36$.

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Table 13:

Marked overpartition (π, j) of 6	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(mod3)
(5+1,1)	5+1	ϕ	1	0	0	1	1
$(\bar{5}+1,1)$	1	5	1	0	0	0	0
$(4+1+1,1)$	4+1+1	ϕ	2	0	1	0	0
$(4+1+1,2)$	4+1+1	ϕ	2	0	0	2	2
$(\bar{4}+1+1,1)$	1+1	4	2	0	1	–1	2
$(\bar{4}+1+1,2)$	1+1	4	2	0	0	1	1
$(3+3,1)$	3+3	ϕ	2	0	1	–1	2
$(3+3,2)$	3+3	ϕ	2	0	0	1	1
$(3+2+1,1)$	3+2+1	ϕ	1	0	0	2	2
$(\bar{3}+2+1,1)$	2+1	3	1	0	0	1	1
$(3+\bar{2}+1,1)$	3+1	2	1	0	0	1	1
$(\bar{3}+\bar{2}+1,1)$	1	3+2	1	0	0	0	0
$(3+1+1+1,1)$	3+1+1+1	ϕ	3	0	2	–1	2
$(3+1+1+1,2)$	3+1+1+1	ϕ	3	0	1	0	0
$(3+1+1+1,3)$	3+1+1+1	ϕ	3	0	0	3	0
$(\bar{3}+1+1+1,1)$	1+1+1	3	3	0	2	–2	1
$(\bar{3}+1+1+1,2)$	1+1+1	3	3	0	1	–1	2
$(\bar{3}+1+1+1,3)$	1+1+1	3	3	0	0	2	2
$(2+2+1+1,1)$	2+2+1+1	ϕ	2	0	1	1	1
$(2+2+1+1,2)$	2+2+1+1	ϕ	2	0	0	3	0
$(\bar{2}+2+1+1,1)$	2+1+1	2	2	0	1	0	0

$(\bar{2}+2+1+1,2)$	2+1+1	2	2	0	0	2	2
(2+1+1+1+1,1)	2+1+1+1+1	ϕ	4	0	3	–3	0
(2+1+1+1+1,2)	2+1+1+1+1	ϕ	4	0	2	–2	1
(2+1+1+1+1,3)	2+1+1+1+1	ϕ	4	0	1	0	0
(2+1+1+1+1,4)	2+1+1+1+1	ϕ	4	0	0	4	1
$(\bar{2}+1+1+1+1,1)$	1+1+1+1	2	4	0	3	–3	0
$(\bar{2}+1+1+1+1,2)$	1+1+1+1	2	4	0	2	–2	1
$(\bar{2}+1+1+1+1,3)$	1+1+1+1	2	4	0	1	–1	2
$(\bar{2}+1+1+1+1,4)$	1+1+1+1	2	4	0	0	3	0
$(1+1+1+1+1+1,1)$	1+1+1+1+1+1	ϕ	6	0	5	–5	1
$(1+1+1+1+1+1,2)$	1+1+1+1+1+1	ϕ	6	0	4	–4	2
$(1+1+1+1+1+1,3)$	1+1+1+1+1+1	ϕ	6	0	3	–3	0
$(1+1+1+1+1+1,4)$	1+1+1+1+1+1	ϕ	6	0	2	–2	1
$(1+1+1+1+1+1,5)$	1+1+1+1+1+1	ϕ	6	0	1	–1	2
$(1+1+1+1+1+1,6)$	1+1+1+1+1+1	ϕ	6	0	0	5	2

We see that the residue of the $\overline{sptcrank}$ (mod 3) divides the marked overpartitions of $3n$ with smallest part not overlined and odd into 3 equal classes. Hence The Corollary.

Corollary 9: The residue of the $\overline{sptcrank}$ (mod 5) divides the marked overpartitions of 5*n* with smallest part not overlined and odd into 5 equal classes.

Proof: We prove the corollary with the help of example. There are 260 marked overpartitions of 5n(when *n* = 2) with smallest part not overlined and odd so that $\overline{spt_1}(10) = 260$.

Table 14:

Marked overpartition (π, j) of 10	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	$(\bmod 5)$
$(\bar{9}+1,1)$	1	9	1	0	0	0	0
$(8+1+1,1)$	8+1+1	ϕ	2	0	1	0	0

$(\bar{7}+3,1)$	3	7	1	0	0	0	0
$(\bar{7}+\bar{2}+1,1)$	1	7+2	1	0	0	0	0
$(7+1+1+1,2)$	7+1+1+1	ϕ	3	0	1	0	0
…	….						
There are 52 marked overpartions with $\overline{sptcrank}$ congruent 0 (mod 5)							

Marked overpartition (π, j) of 10	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(mod 5)
$(9+1,1)$	9+1	ϕ	1	0	0	1	1
$(\bar{8}+1+1,2)$	1+1	8	2	0	0	1	1
$(7+3,1)$	7+3	ϕ	1	0	0	1	1
$(\bar{7}+2+1,1)$	2+1	7	1	0	0	1	1
$(7+\bar{2}+1,1)$	7+1	2	1	0	0	1	1
...	….						
There are 52 marked overpartitions with $\overline{sptcrank}$ congruent 1 (mod 5)							

Marked overpartition (π, j) of 10	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(mod 5)
(8+1+1, 2)	8+1+1	ϕ	2	0	0	2	2
(7+2+1,1)	7+2+1	7	1	0	0	2	2
$(\bar{7}+1+1+1,3)$	1+1+1	7	3	0	0	2	2
(6+3+1,1)	6+3+1	ϕ	1	0	0	2	2
$(\bar{6}+2+1+1,2)$	2+1+1	6	2	0	0	2	2
...	….						
There are 52 marked overpartitions with $\overline{sptcrank}$ congruent 2 (mod 5)							

Marked overpartition (π, j) of 10	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(mod 5)
(7+1+1+1, 3)	7+1+1+1	ϕ	3	0	0	3	3
$(\bar{7}+1+1+1,1)$	1+1+1	7	3	0	2	–2	3
(6+2+1+1,2)	6+2+1+1	ϕ	2	0	0	3	3
$(\bar{6}+1+1+1+1,2)$	1+1+1+1	6	4	0	2	–2	3
$(\bar{6}+1+1+1+1,4)$	1+1+1+1	6	4	0	0	3	3
...	…..						
There are 52 marked overpartitions with $\overline{sptcrank}$ congruent 3 (mod 5)							

Marked overpartition (π, j) of 10	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(mod 5)
$(\bar{8}+1+1,1)$	1+1	8	2	0	1	–1	4
$(7+1+1+1,1)$	7+1+1+1	ϕ	3	0	2	–1	4
$(\bar{7}+1+1+1,2)$	1+1+1	7	3	0	1	–1	4
$(\bar{6}+\bar{2}+1+1,1)$	1+1	6+2	2	0	1	–1	4
(6+1+1+1+1,1)	6+1+1+1+1	ϕ	3	0	2	–1	4
...	….						
There are 52 marked overpartitions with $\overline{\text{sptcrank}}$ congruent 4 (mod 5)							

We see that the residue of the $\overline{\text{sptcrank}}$(mod 5) divides the marked overpartitions of 5*n* with smallest part not overlined and odd into 5 equal classes. Hence The Corollary.

Corollary 10 [9]: The residue of the $\overline{sptcrank}(\text{mod}\,3)$ divides the marked overpartitions of 3*n* with the smallest part not overlined and even into 3 equal classes.

Proof: We prove the Corollary with the help of an example. There are 6 marked overpartitions of 3*n* (when *n* = 2) with the smallest part not overlined and even so that, $\overline{spt}_2(6) = 6$.

Table 15:

Marked overpartition (π, j) of 6	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(mod 3)
(6 ,1)	6	ϕ	1	0	0	0	0
(4+2 ,1)	4+2	ϕ	1	0	0	1	1
$(\bar{4}+2,1)$	2	4	1	0	0	0	0
(2+2+2, 1)	2+2+2	ϕ	3	0	2	–2	1
(2+2+2, 2)	2+2+2	ϕ	3	0	1	–1	2
(2+2+2, 3)	2+2+2	ϕ	3	0	0	2	2

We see that the residue of the $\overline{sptcrank}(\text{mod}\,3)$ divides the marked overpartitions of 3*n* (when *n* = 2) with smallest part not overlined and even into 3 equal classes. Hence The Corollary.

Corollary 11: The residue of the $\overline{sptcrank}(\text{mod}\,3)$ divides the marked overpartitions of 3*n*+1 with smallest part not overlined and even into 3 equal classes.

Proof: We prove the Corollary with the help of an example. There are 6 marked overpartitions of 3n+1(when *n* = 2) with the smallest part not overlined and even, so that $\overline{spt_2}(7) = 6$.

Table 16:

Marked overpartition (π, j) of 7	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(mod 3)
(5+2, 1)	5+2	ϕ	1	0	0	1	1
$(\bar{5}+2,1)$	2	5	1	0	0	0	0
(3+2+2, 1)	3+2+2	ϕ	2	0	1	0	0
(3+2+2, 2)	3+2+2	ϕ	2	0	0	2	2
$(\bar{3}$+2+2, 1)	2+2	3	2	1	2	–2	1
$(\bar{3}$+2+2, 2)	2+2	3	2	1	1	–1	2

We see that the residue of the $\overline{sptcrank}$ (mod 3) divides the marked overpartitions of 3*n*+1 with smallest part not overlined and even. Hence The Corollary.

Corollary 12: The residue of the $\overline{sptcrank}(\text{mod}\,5)$ divides the marked overpartitions of 5*n*+3 with smallest part not overlined and even into 5 equal classes.

Proof: We prove the Corollary with the help of example. There are 15 marked overpartitions of 5*n* + 3 (when *n* =1) with the smallest part not overlined and even so that $\overline{spt_2}(8) = 15$.

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Table 17:

Marked overpartition (π, j) of 8	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(mod 5)
$(\bar{6}+2,1)$	2	6	1	2	2	–2	3
$(\bar{4}+2+2,1)$	2+2	4	2	0	1	–1	4
$(\bar{4}+2+2,2)$	2+2	4	2	0	0	1	1
$(\bar{3}+3+2,1)$	3+2	3	1	1	1	0	0
(2+2+2+2, 1)	2+2+2+2	ϕ	4	0	3	–3	2
(2+2+2+2, 2)	2+2+2+2	ϕ	4	0	2	–2	3
(2+2+2+2, 3)	2+2+2+2	ϕ	4	0	1	–1	4
(2+2+2+2, 4)	2+2+2+2	ϕ	4	0	0	3	3
(3+3+2, 1)	3+3+2	ϕ	1	0	0	2	2
(4+2+2, 1)	4+2+2	ϕ	1	0	1	0	0
(4+2+2, 2)	4+2+2	ϕ	2	0	0	2	2
(6+2, 1)	6+2	ϕ	1	0	0	1	1
(4+4, 1)	4+4	ϕ	2	0	1	–1	4
(4+4, 2)	4+4	ϕ	2	0	0	1	1
(8, 1)	8	ϕ	1	0	0	0	0

We see that the residue of the $\overline{sptcrank}(\text{mod } 5)$ divides the marked overpartitions of of 5*n* + 3 with the smallest part not overlined and even into 5 equal classes. Hence The corollary.

5. CONCLUSION

In this study we have found the number of smallest parts in the overpartitions of *n* with smallest part not overlined, not overlined and odd, not overlined and even for *n*=1,2,3,4,… respectively. We have shown the relations $\overline{spt}\,(3n) \equiv 0$ (mod3), $\overline{spt}_1(3n) \equiv 0$ (mod3) for $n \geq 0$, $\overline{spt}_1(n) \equiv 1$ (mod 2) if *n* is an odd square, $\overline{spt}_1(5n) \equiv 0$ (mod5), $\overline{spt}_2(3n) \equiv 0(\text{mod } 3)$, $\overline{spt}_2(3n+1) \equiv 0(\text{mod } 3)$ and $\overline{spt}_2(5n+3) \equiv 0$ (mod 5) with the help of induction method and have shown the some results with the help of vector partitions along with their weights and cranks . We have verified the Theorems for certain values of n and have established the some Corollaries with the help of marked overpartitions.

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