DEVELOPMENT OF ANDREWS-GARVAN-LIANG’S SELF-CONJUGATE S-PARTITIONS

Sabuj Das*
* Senior Lecturer, Department of Mathematics, Raozan University College, BANGLADESH
*Correspondence Author: sabujdas.ctg@gmail.com

Abstract:
In 2013, Andrews, Garvan and Liang defined Self-conjugate S-partitions. In 2011, Andrews stated the definition of spt(n). This paper shows how to find the Self-conjugate S-partitions of 4 and 5 respectively, and proves the Corollary-1 that is ‘The number of Self-conjugate S-partitions counted according to the weight w is congruent to the total number of smallest parts in all the partitions of n modulo 2’. This paper shows how to generate the generating functions for $M_{sc}(n)$ in different ways. This paper shows how to find the number of partitions of n with an odd number of smallest parts and a total number of even (respectively odd) parts, and shows how to find the number of partitions of n in odd parts without gaps, and also shows how to find the number of partitions of n in odd parts without gaps, and also shows how to generate the generating functions for $(A_e(n) - A_c(n))$ and $(L_1(n) - L_3(n))$ respectively. This paper shows how to prove the further three Corollaries with the help of examples, and shows how to prove the three Theorems by easy algebraic method.

Keywords:
Crank, congruent to involution, product notations, Self-conjugate, spt(n), weight.

1. INTRODUCTION

We give some related definitions of $P(n)$, Self-conjugate S-partitions, $M_{sc}(n)$, $A_e(n)$, $A_o(n)$, $L_1(n)$, $L_3(n)$, $(x)_\infty$, $(x;x^2)_\infty$, $(x^2;x^2)_\infty$, and spt(n). We give two tables of the Self-conjugate S-partitions of 4 and 5 respectively and introduce Corollary-1 in terms of $M_{sc}(n)$ and spt(n). We discuss the various generating functions for $M_{sc}(n)$, $(A_o(n) - A_c(n))$ and $(L_1(n) - L_3(n))$ and give some tables of the partitions of n with an odd number of smallest parts and of the partitions of n in odd parts without gaps. We discuss the number of Self-conjugate S-partitions counted according the weight $w$, and give further three Corollaries in terms of $(A_o(n) - A_c(n))$, $(L_1(n) - L_3(n))$ and $M_{sc}(n)$. Finally we prove the three Theorems with the help of various generating functions.

2. SOME RELATED DEFINITIONS

$P(n)$[7]:The number of partitions of n like 4, 3+1, 2+2, 2+1+1, 1+1+1+1. $\therefore P(4) = 5$

Self-conjugate S-partitions [3,5]:
Let D denote the set of partitions into distinct parts and P denote the set of partitions. The set of vector partitions V is defined the Cartesian product

V = D × P × P. If S is the subset of V,

\[ S = \{ \pi = (\pi_1, \pi_2, \pi_3) \in V : 1 \leq (\pi_j) < \infty \text{ and } s(\pi_j) \leq \min\{s(\pi_2), s(\pi_3)\} \}. \]

Here \( s(\pi) \) is the smallest part in the partition with the convention that \( s(\phi) = \infty \) for the empty partition. We call the vector partitions in S simply S-partitions for \( \pi = (\pi_1, \pi_2, \pi_3) \in s \), we define the weight \( w(\pi) = (-1)^{#(\pi_1)} \), the Crank \( (\pi) = \#(\pi_2) - \#(\pi_3) \) and \( |\pi| = |\pi_1| + |\pi_2| + |\pi_3| \). When \( |\pi_1| \) is the sum of the parts of \( \pi_1 \) and \# \( \pi_j \) denotes the number of parts of \( \pi_j \).

The map \( T : S \rightarrow S \) given by,

\[ T(\pi) = T(\pi_1, \pi_2, \pi_3) = T(\pi_1, \pi_3, \pi_2) \]

is natural involution. An S-partition \( \pi = (\pi_1, \pi_3, \pi_2) \) is a fixed point of \( T \) if and only if \( \pi_2 = \pi_3 \). We call these fixed points “Self-conjugate S-partitions”.

The number of Self-conjugate S-partitions counted according to the weight \( w \) is denoted by \( M_{sc}(n) \), So that,

\[ M_{sc}(n) = \sum w(\pi) \]

\[ \pi \in S, \quad |\pi| = n \]

\[ T(\pi) = \pi \]

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\[ M_{sc}(n) = \sum w(\pi) \]

\[ \pi \in S, \quad |\pi| = n \]

\[ T(\pi) = \pi \]

\( A_e(n) \): The number of partitions of \( n \) with an odd number of smallest parts and a total number of even parts.

\( A_o(n) \): The number of partitions of \( n \) with an odd number of smallest parts, and a total number of odd parts.

\( L_1(n) \): The number of partitions of \( n \) in odd parts with no gaps, and the largest part is congruent to 1 mod 4.

\( L_3(n) \): The number of partitions of \( n \) in odd parts with no gaps, and the largest part is congruent to 3 mod 4.
Product notations [6]:

\((x)_{\infty} = (1-x)(1-x^2)(1-x^3)\ldots\)

\((x;x^2)_{\infty} = (1-x)(1-x^3)(1-x^5)\ldots\)

\((x^2;x^2)_{\infty} = (1-x^2)(1-x^4)(1-x^6)\ldots\)

\((-x;x)_{\infty} = (1+x)(1+x^2)(1+x^3)\ldots\)

\(\text{spt}(n) : \text{spt}(n) \) The total numbers of appearances of the smallest parts in all the partitions of \(n\) like,

\[
\begin{array}{c|c}
\text{n} & \text{spt(n)} \\
1 & 1 \\
2 & 3 \\
3 & 5 \\
4 & 10 \\
\end{array}
\]

3. THERE ARE TWO TABLES OF THE SELF-CONJUGATE S-PARTITIONS

OF 4 AND 5: We get;

\[\text{Table-1}\]

<table>
<thead>
<tr>
<th>Self-conjugate S-partition of 4</th>
<th>Weight (w(\pi))</th>
<th>Crank (\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi_1 = (4,\phi,\phi))</td>
<td>+ 1</td>
<td>0</td>
</tr>
<tr>
<td>(\pi_2 = (3+1,\phi,\phi))</td>
<td>- 1</td>
<td>0</td>
</tr>
<tr>
<td>(\sum w(\pi) = 0)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(\therefore M_\infty(4) = 0\). Here we have used \(\phi\) to indicate the empty partition. Again;

\[\text{Table-2}\]

<table>
<thead>
<tr>
<th>Self-conjugate S-partition of 5</th>
<th>Weight (w(\pi))</th>
<th>Crank (\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi_1 = (5,\phi,\phi))</td>
<td>+ 1</td>
<td>0</td>
</tr>
<tr>
<td>(\pi_2 = (1,2,2))</td>
<td>+ 1</td>
<td>0</td>
</tr>
<tr>
<td>(\pi_3 = (1,1+1,1+1))</td>
<td>+ 1</td>
<td>0</td>
</tr>
<tr>
<td>(\pi_4 = (1+4,\phi,\phi))</td>
<td>- 1</td>
<td>0</td>
</tr>
</tbody>
</table>
\[ \pi_5 = (2 + 3, \phi, \phi) \]
\[ \pi_6 = (2 + 1, 1,1) \]
\[ \sum w(\pi) = 0 \]

\[ M_{\text{sc}}(5) = \sum_{i=1}^6 w(\pi_i) = 1 + 1 + 1 = 0. \]

**Corollary-1** \( M_{\text{sc}}(n) \equiv spt(n)(\text{mod } 2) \)

**Proof:** From above table we get;
\[ M_{\text{sc}}(1) = 1, M_{\text{sc}}(2) = 1, M_{\text{sc}}(3) = 1, \ldots \]
\[ M_{\text{sc}}(3) = 1 \equiv 1(\text{mod } 2), spt(3) = 5 \equiv 1(\text{mod } 2) \]
\[ M_{\text{sc}}(4) = 0 \equiv 0(\text{mod } 2), spt(4) = 10 \equiv 0(\text{mod } 2) \]

We can conclude that,
\[ M_{\text{sc}}(n) \equiv spt(n)(\text{mod } 2). \] Hence the Corollary.

4. **NOW WE DESCRIBE THE GENERATING FUNCTIONS**

The generation functions for \( M_{\text{sc}}(n) \) are given by

\[ \sum_{n=1}^\infty \frac{x^n(x^{n+1};x)_\infty}{(x^{2n};x^2)_\infty} = \frac{x(x^2;x)_\infty}{(x^2;x^2)_\infty} + \frac{x^2(x^3;x)_\infty}{(x^4;x^4)_\infty} + \frac{x^3(x^4;x)_\infty}{(x^6;x^6)_\infty} \]

\[ = (x - x^4 - x^6 - \ldots) + (x^2 - x^5 - x^7 - \ldots) + (x^3 - x^7 + x^4 + x^5 + \ldots) \]

\[ = (x + x^2 + x^3 + 0.x^4 + 0.x^5 + \ldots) \]

\[ = M_{\text{sc}}(1)x + M_{\text{sc}}(2)x^2 + M_{\text{sc}}(3)x^3 + M_{\text{sc}}(4)x^4 + M_{\text{sc}}(5)x^5 + \ldots \]

\[ = \sum_{n=1}^\infty M_{\text{sc}}(n)x^n. \quad \therefore \sum_{n=1}^\infty M_{\text{sc}}(n)x^n = \sum_{n=1}^\infty \frac{x^n(x^{n+1};x)_\infty}{(x^{2n};x^2)_\infty}. \]

Again, we get;

\[ \frac{1}{(1-x)} \sum_{n=1}^\infty \frac{x^n(-x;x)_\infty}{(1-x^n)} \]

\[ = \frac{1}{(1+x)(1+x^2)\ldots} \left\{ \frac{x}{1-x} + \frac{x^2(1+x)}{1-x^2} + \frac{x^3(1+x)(1+x^2)}{1-x^3} + \ldots \right\} \]

\[ = (1-x-x^3+x^4-x^5+x^7+\ldots)(x+x^3+x^4+x^5+x^6+\ldots+x^2+x^3+x^4+\ldots) \]

\[ = (1-x-x^3+x^4-x^5)(x+2x^2+3x^3+4x^4+5x^5+\ldots) \]

\[ = x+x^2+x^3+0.x^4+0.x^5+\ldots \]

\[ = M_{\text{sc}}(1)x + M_{\text{sc}}(2)x^2 + M_{\text{sc}}(3)x^3 + M_{\text{sc}}(4)x^4 + M_{\text{sc}}(5)x^5 + \ldots \]
\[= \sum_{n=1}^{\infty} M_{sc} (n)x^n. \]
\[
\therefore \sum_{n=1}^{\infty} M_{sc} (n)x^n = \frac{1}{(-x;x)_\infty} \sum_{n=1}^{\infty} \frac{x^n(-x;x)_{n-1}}{(1-x^n)}. \]

Again we get;
\[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{(x;x^2)_n} = \frac{x}{(1-x)(1-x^3)} + \ldots. \]
\[= x\left(1 + x + x^2 + x^3 + x^4 + x^5\right) - x^4(1 + x + x^2 + \ldots)(1 + x^3 + \ldots) \]
\[= x + x^2 + x^3 + 0.x^4 + 0.x^5 + \ldots. \]
\[= M_{sc}(1)x + M_{sc}(2)x^2 + M_{sc}(3)x^3 + M_{sc}(4)x^4 + M_{sc}(5)x^5 + \ldots. \]
\[= \sum_{n=1}^{\infty} M_{sc} (n)x^n. \therefore \sum_{n=1}^{\infty} M_{sc} (n)x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{(x;x^2)_n} \]

Also we get;
\[\sum_{n=0}^{\infty} \frac{1}{(x^2;x^2)_n} \{(x)_{2n} - (x)_\infty\} \]
\[= \left\{1 - (1-x)(1-x^2) \ldots \right\} + \frac{\left\{(1-x)(1-x^2) - (1-x)(1-x^2) \ldots \right\}}{1 - x^2} \]
\[+ \frac{\left\{(1-x)(1-x^2)(1-x^3)(1-x^4) - (1-x)(1-x^2)(1-x^3) \ldots \right\}}{(1-x^2)(1-x^4)} + \ldots \]
\[= x + x^2 + x^3 + 0.x^4 + 0.x^5 + \ldots. \]
\[= M_{sc}(1)x + M_{sc}(2)x^2 + M_{sc}(3)x^3 + M_{sc}(4)x^4 + M_{sc}(5)x^5 + \ldots. \]
\[= \sum_{n=1}^{\infty} M_{sc} (n)x^n. \therefore \sum_{n=1}^{\infty} M_{sc} (n)x^n = \sum_{n=0}^{\infty} \frac{1}{(x^2;x^2)_n} \{(x)_{2n} - (x)_\infty\}. \]

The partitions of 5 with an odd number of smallest parts are in the table

<table>
<thead>
<tr>
<th>Partition ((\pi) of 5)</th>
<th># ((\pi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>4+1</td>
<td>2</td>
</tr>
<tr>
<td>3+2</td>
<td>2</td>
</tr>
<tr>
<td>2+2+1</td>
<td>3</td>
</tr>
<tr>
<td>2+1+1+1</td>
<td>4</td>
</tr>
<tr>
<td>1+1+1+1+1</td>
<td>5</td>
</tr>
</tbody>
</table>

We see that \(A_5(5)=3\), and \(A_6(5)=3\). \therefore \(A_5(5) - A_6(5)=0\).

The partitions of 6 with an odd number of smallest parts are in the table

<table>
<thead>
<tr>
<th>Partition ((\pi) of 6)</th>
<th># ((\pi))</th>
</tr>
</thead>
</table>

We see that \( A_0(6)=3, \ A_e(6)=3. \quad \therefore A_0(6) - A_e(6)=0, \)

Similar, we get;
\[ A_0(1)- A_e(1)=1- 0=1 \]
\[ A_0(2)- A_e(2)=1- 0=1 \]
\[ A_0(3)- A_e(3)=1- 0=1 \]
\[ A_0(4)- A_e(4)=1-0=1 \]

The generating function for \( \left(A_0(n) - A_e(n)\right) \) is given by
\[
\sum_{n=1}^{\infty} \frac{x^n}{(1-x^2n)(-x^{n+1};x)_{\infty}} = \frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^3}{1-x^6} + \ldots
\]
\[= x + x^2 + x^3 + 0.x^4 + 0.x^5 + 0.x^6 + \ldots \]
\[= \{A_0(1) - A_e(1)\}x + \{A_0(2) - A_e(2)\}x^2 + \{A_0(3) - A_e(3)\}x^3 + \ldots \]
\[= \sum_{n=1}^{\infty} \{A_0(n) - A_e(n)\}x^n. \]

**Corollary-2:** \( A_0(n) - A_e(n) = M_{sc}(n) \)

Proof: We get, the generating function for \( (A_0(n) - A_e(n)) \) is
\[
\sum_{n=1}^{\infty} \left(A_0(n) - A_e(n)\right) x^n = \sum_{n=1}^{\infty} \left(\frac{x^n}{1-x^{2n}} \cdot \frac{1}{(-x^{n+1};x)_{\infty}} \right)
\]
\[= \frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^3}{1-x^6} + \ldots
\]
\[= \frac{x}{1-x(1+x)} + \frac{x^2}{(1-x^2)(1+x)(1+x^2)} + \frac{x^3}{(1-x^3)(1+x)(1+x^2)} + \ldots
\]
\[= \frac{1}{(1+x)(1+x^2)(1+x^3)} \left\{ \frac{x}{1-x^2} + \frac{x^2(1+x)}{1-x^2} + \frac{x^3(1+x)(1+x^2)}{1-x^3} + \ldots \right\}
\]
\[= \frac{1}{(-x;x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n(-x;x)_{n-1}}{1-x^n} = \sum_{n=1}^{\infty} M_{sc}(n)x^n
\]

Equation the co-efficient of \( x^n \) from both sides we get;
\[ A_0(n) - A_e(n) = M_{sc}(n). \] Hence the Corollary .

The partitions of 5 in odd parts with no gaps are in the table

\[
\begin{align*}
\text{Table-5} \\
\text{Partition (\(\pi\)) of 5} & \quad \text{Largest part} \\
3+1+1 & \quad 3 \\
1+1+1+1+1 & \quad 1
\end{align*}
\]

We see that \( L_1(5) = 1, \) \( L_3(5) = 1. \) \( \therefore L_1(5) - L_3(5) = 1 - 1 = 0. \)

The partitions of 6 in odd parts with no gaps are in the table

\[
\begin{align*}
\text{Table-6} \\
\text{Partition (\(\pi\)) of 6} & \quad \text{Largest part} \\
3+1+1+1 & \quad 3 \\
1+1+1+1+1 & \quad 1
\end{align*}
\]

We set that \( L_1(6) = 1, \) \( L_3(6) = 1. \) \( \therefore L_1(6) - L_3(6) = 1 - 1 = 0. \)

Similarly we get;

\[
\begin{align*}
L_1(1) - L_3(1) & = 1 - 0 = 1 \\
L_1(2) - L_3(2) & = 1 - 0 = 1 \\
L_1(3) - L_3(3) & = 1 - 0 = 1 \\
L_1(4) - L_3(4) & = 1 - 1 = 0
\end{align*}
\]

The generating function for \( (L_1(n) - L_3(n)) \) is given by

\[
\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(1-x)(1-x^3)} = x - \frac{x^4}{1-x} + \ldots
\]

\[
n = x + x^2 + x^3 + x^4 + x^5 + \ldots - x^4 (1 + x + x^2 + \ldots)(1 + x^3 + \ldots) + \ldots
\]

\[
= x + x^2 + x^3 + 0.x^4 + 0.x^5 + 0.x^6 - x^7 + \ldots
\]

\[
= \{L_1(1) - L_3(1)\}x + \{L_1(2) - L_3(2)\}x^2 + \{L_1(3) - L_3(3)\}x^3 + \ldots
\]

\[
= \sum_{n=1}^{\infty} L_1(n) - L_3(n)x^n.
\]

\textbf{Corollary-3} \( A_0(n) - A_e(n) = L_1(n) - L_3(n) = M_{sc}(n) \)

Proof: From above we get; \( A_0(1) - A_e(1) = 1 = L_1(1) - L_3(1) \) and \( M_{sc}(1) = 1\)

\( A_0(2) - A_e(2) = 1 = L_1(2) - L_3(2) \) and \( M_{sc}(2) = 1 \)

\( A_0(3) - A_e(3) = 1 = L_1(3) - L_3(3) \) and \( M_{sc}(3) = 1 \)

We can conclude that,

\( A_0(n) - A_e(n) = L_1(n) - L_3(n) = M_{sc}(n). \) Hence the Corollary .
Corollary-4  \( A_0(n) - A_e(n) = L_1(n) - L_3(n) \equiv \text{spt}(n) \pmod{2} \)
\( A_0(1) - A_e(1) = L_1(1) - L_3(1) = 1 \equiv 1 \pmod{2} \) and \( \text{spt}(1) = 1 \equiv 1 \pmod{2} \)
\( A_0(4) - A_e(4) = L_1(4) - L_3(4) = 0 \equiv 0 \pmod{2} \) and \( \text{spt}(4) = 10 \equiv 1 \pmod{2} \)
\( A_0(6) - A_e(6) = L_1(6) - L_3(6) = 0 \equiv 0 \pmod{2} \) and \( \text{spt}(6) = 26 \equiv 0 \pmod{2} \)

We can conclude that,
\( A_0(n) - A_e(n) = L_1(n) - L_3(n) \equiv \text{spt}(n) \pmod{2} \). Hence the Corollary.

Theorem 1: \( \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} = \frac{1}{(-x; x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{(1-x^n)} \)

Proof: Left hand side = \( \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} \cdot \frac{1}{(-x^{n+1}; x)_{\infty}} \)

\begin{align*}
= \frac{x}{(1-x^2)} &+ \frac{x^2(1+x)}{(1-x^2)(1+x^3)} + \frac{x^3(1+x)(1+x^2)}{(1-x^2)(1+x^3)(1+x^4)} + \cdots \\
= \frac{1}{(1-x)(1+x)(1+x^2)} &+ \frac{x^2(1+x)}{(1-x^2)(1+x^3)} + \frac{x^3(1+x)(1+x^2)}{(1-x^2)(1+x^3)(1+x^4)} + \cdots \\
= \frac{1}{(1-x)(1+x)(1+x^2)(1+x^3)} \left[ \frac{x}{1-x} + \frac{x^2(1+x)}{1-x^2} + \frac{x^3(1+x)(1+x^2)}{1-x^3} + \cdots \right] \\
= \frac{1}{(-x; x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{1-x^n} = \text{Right hand side. Hence the Theorem.} \\
\end{align*}

Theorem 2: \( \frac{1}{(-x; x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{(1-x^n)} = \sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_{n}} \{((x)_{2n} - (x)_{x})\} \)

Proof: We get;
\begin{align*}
= \frac{1}{(1-x^{2})(1-x^{4})} &+ \frac{x^2(1+x)}{(1-x)(1-x^2)(1-x^4)} + \frac{x^3(1-x^2)(1-x^4)}{(1-x)(1-x^2)(1-x^4)(1-x^6)} + \cdots \\
= \frac{1}{(1-x)(1-x^2)(1-x^4)} &+ \frac{x^2}{(1-x)(1-x^2)(1-x^4)} + \frac{x^3}{(1-x)(1-x^2)(1-x^4)(1-x^6)} + \cdots \\
\end{align*}
\[\sum_{n=1}^{\infty} \frac{x^n}{(x)^n(x^2; x^2)} = \sum_{n=0}^{\infty} \frac{x^n}{(x^2; x^2)} \sum_{k=0}^{\infty} \frac{x^{2nk}}{(x^{2n}; x^2)} \]  

\[[1], P.19\]

\[= \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} \sum_{n=1}^{\infty} \frac{x^n}{(x)^n} = \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} \left( \frac{x^{2k+1}}{(1-x)} + \frac{x^{2(2k+1)}}{(1-x)(1-x^2)} + \ldots \right)\]

\[= \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} \left[ x^{2k+1} + x^{2k+2} + x^{2k+3} + \ldots \right] = \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} \left[ x^{2k+1} + x^{2k+2} + x^{2k+3} + \ldots - 1 \right] = \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} \left[ \frac{1}{1-x^{2k+1}} \frac{1}{1-x^{2k+2}} \ldots - 1 \right]

Multiplying both sides by \((x)^\infty\) we have:

\[\frac{(x)^\infty}{(x^2; x^2)^\infty} \sum_{n=1}^{\infty} \frac{x^n}{(-x; x)_{n-1}} = \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} \left[ \frac{(x)^\infty}{(1-x^{2k+1})} \frac{(x)^\infty}{(1-x^{2k+2})} - \frac{(x)^\infty}{(1-x^n)^n} \right] = \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} \left[ (x)_{2k} - (x)^\infty \right]

\[\text{Since} \quad \frac{(x)^\infty}{(x^2; x^2)^\infty} = \frac{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots}{(1-x^2)(1-x^4)(1-x^6)\ldots} = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots} = \frac{1}{(-x; x)^\infty}\]

And

\[\frac{1}{(-x; x)^\infty} = \frac{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots}{(1-x^2)(1-x^4)(1-x^6)\ldots} + \ldots = \frac{1}{(-x; x)^\infty}
\]

\[= 1 + (1-x)(1-x^2) + (1-x)(1-x^2)(1-x^3)(1-x^4) + \ldots = \sum_{k=0}^{\infty} (x)_{2k}\]

\[\sum_{n=1}^{\infty} \frac{x^n}{(-x; x)_{n-1}} = \sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} \left[ (x)^{2n} - (x)^\infty \right].\]

\[\therefore \text{Left hand side} = \sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} \left[ (x)^{2n} - (x)^\infty \right] = \text{Right hand side. Hence the Theorem.}\]

**Theorem 3:** \[\sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} \left[ (x)^{2n} - (x)^\infty \right] = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}x^{2n}}{(x; x^2)_n}\]

**Proof:** Left hand side = \[\sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} \left[ (x; x)_{2n} - (x; x)^\infty \right]
\]

\[= \sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} \left[ (x; x)_{2n} - (x; x)^\infty \right] = \sum_{n=0}^{\infty} \left[ (x; x)^n - (x; x)^\infty \right] \]
[Since \( \frac{(1-x)(1-x^2)(1-x^3)(1-x^4)}{(1-x^2)(1-x^4)} \)....

\( = \frac{(1-x)(1-x^2)}{(1-x^2)(1-x^4)} \)....

\( = \sum_{n=0}^{\infty} \left[ (x; x^2)_n - (x; x^2)_{\infty} + (x; x^2)_n - \frac{(x; x)_{\infty}}{(x^2; x^2)_n} \right] \]

\( = \sum_{n=0}^{\infty} \left[ [(x; x^2)_n - (x; x^2)_{\infty}] + [(x; x^2)_n - \frac{(x; x)_{\infty}}{(x^2; x^2)_n}] \right] \]

\( = -\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(x^2; x^2)_n} + \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} + \sum_{n=0}^{\infty} \left\{ \frac{(x; x)_{\infty}}{(x^2; x^2)_{\infty}} - \frac{(x; x)_{\infty}}{(x^2; x^2)_n} \right\} \) [4]

with \( x \to x^2 \), \( a \to 0 \) and \( t = x \)

\( = -\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(x; x^2)_n} + \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} + \sum_{n=0}^{\infty} \left\{ \frac{1}{(x^2; x^2)_{\infty}} - \frac{1}{(x^2; x^2)_n} \right\} \)

\( = -\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(x; x^2)_n} + \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} + \sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_{\infty}} \sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}} \) [4]

with \( x \to x^2 \), \( a = b = c = 0 \)

\( = -\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(x; x^2)_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n-1}}{(x; x^2)_n} \) = Right hand side. Hence the Theorem.

5. CONCLUSION
We have shown all Self-conjugate S-partitions of 4 and 5 respectively, and have satisfied the Corollary-1 \( M_{sc}(n) \equiv spt(n)(\text{mod} 2) \) for any positive integral value of \( n \). We have shown the partitions of \( n \) with an odd number of smallest parts for \( n = 5 \) and 6, and also have found the partitions of \( n \) in odd parts with no gaps for any positive integral value of \( n \). We have introduced further three Corollaries in terms of \( M_{sc}(n) \) and \( spt(n) \) respectively and have proved three Theorems of Self-conjugate S-partitions with the help of various generating functions.

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7. REFERENCES


[4] G E Andrews J. Jimenez-Urroz, and K.Ono, q-series identities and values of certain L-

