THE CORDIAL LABELING FOR THE CARTESIAN PRODUCT BETWEEN PATHS AND CYCLES

Mohammed M. Ali Al-Shamiri 1,2, A. Elrokh 3, Yasser El–Mashtawy 1, S. Euat Tallah 3
1 Department of Mathematics, Faculty of Science and Arts, Mahayil Asir, King Khalid University, K.S.A.
2 Department of Mathematics and Computer, Faculty of Science, Ibb University, Ibb, Yemen
3 Department of Mathematics, Faculty of Science, Menoufia University, Shebeen Elkom, Egypt

Abstract

A graph is said to be cordial if it has a 0-1 labeling that satisfies certain properties. In this paper we show the Cartesian product of a path and a cycle or vice versa are always cordial under some conditions. Also, we prove that the Cartesian product of two paths is cordial.

Keywords: Cartesian Product; Cordial; Cycle; Graph; Labeling; Path.


1. Introduction

It is known that graph theory and its branches have become interest topics for almost all fields of mathematics and also other area of science such as chemistry, biology, physics, communication, economics, and computer science. A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. There are many contributions and different kinds of labeling [6-10,11]. Suppose that $G = (V, E)$ is a graph, where $V$ is the set of its vertices and $E$ is the set of its edges. Throughout, it is assumed $G$ is connected, finite, simple and undirected. A binary vertex labeling of $G$ is a mapping $f : V \rightarrow \{0,1\}$ in which $f(u)$ is said to be the labeling of $u \in V$. For an edge $e = uv \in E$, where $u,v \in V$, the introduced edge labeling $f^*(e) : E \rightarrow \{0,1\}$ is defined by the formula $f^*(uv) = |f(v) + f(w)| \pmod{2}$. Thus, for any edge $e$, $f^*(e) = 0$ if its two vertices have the same labeling and $f^*(e) = 1$ if they have different labeling. Let us denote $V_0$ and $V_1$ be the numbers of vertices labeled by 0 and 1 in $V$ respectively, and let $E_0$ and $E_1$ be the corresponding numbers of edges in $E$ labeled by 0 and 1 respectively. A binary vertex labeling $f$ of $G$ is said to be cordial if $|V_0 - V_1| \leq 1$ and $|E_0 - E_1| \leq 1$. A graph $G$ is cordial if it has cordial labeling. The cordial graphs were introduced by Cahit [1] as a weaker version of both graceful and harmonious graphs [7-9]. A lot of operations on graphs have been applied on the cordial
labeling[2-5]. The Cartesian product of two graphs is that Boolean operation $G = G_1 \times G_2$ in which for any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$, the line $uv$ is in $X$ whenever $[u_1 = v_1 \text{ and } u_2 v_2 \in X_2]$ or $[u_2 = v_2 \text{ and } u_1 v_1 \in X_1]$ where $X, X_1, X_2$ are the sets of edges of $G, G_1 \text{ and } G_2$ respectively [12]. It follows from the definition of the Cartesian product that the graph $G_1 \times G_2$ has $n_1 n_2 \text{ vertices and } n_1 m_2 + n_2 m_1 \text{ edges where } G_1$ has $n_1, m_1 \text{ vertices and edges and } G_2$ has $n_2, m_2 \text{ vertices and edges.}$ In this paper we show that $P_n \times C_m, C_n \times P_m \text{ and } P_n \times P_m$ are cordial for some $n$ and $m$.

2. Terminologies and Notation

A path with $n$ vertices and $n - 1$ edges is denoted by $P_n$, and a cycle with $n$ vertices and $n$ edges is denoted by $C_n$. Given a path or a cycle with $4r$ vertices, we let $L_{4r}$ denote the labeling $0011\ldots 0011$ (repeated $r$-times). In most cases, we modify this by adding symbols at one end or the other (or both); thus $O_{4r}$ denotes the labeling $010011\ldots 0011$ of the path $P_{4r+3}$ (or the cycle $C_{4r+3}$) when $r \geq 1$ and $010$ when $r = 0$, and so on [3-6]. We let $O_r$ denote the labeling $00\ldots 0\text{ (zero repeated } r \text{ times)}, 1_r \text{ denote the labeling } 11\ldots 1\text{ (one repeated } r \text{ times)}, M_r \text{ denotes the labeling } 0101\ldots 011\text{ if } r \text{ is even and } 0101\ldots 010\text{ if } r \text{ is odd.}$ The Cartesian product of two graphs is that Boolean operation $G = G_1 \times G_2$ in which for any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$, the edge $uv$ is in $X$ whenever $[u_1 = v_1 \text{ and } u_2 v_2 \in X_2]$ or $[u_2 = v_2 \text{ and } u_1 v_1 \in X_1]$ where $X, X_1, X_2$ are the sets of edges of $G, G_1 \text{ and } G_2$ respectively [12]. See Figure 2.1. It follows from the definition of the Cartesian product that the graph $G_1 \times G_2$ has $n_1 n_2 \text{ vertices and } n_1 m_2 + n_2 m_1 \text{ edges where } G_1$ has $n_1, m_1 \text{ vertices and edges, } G_2 \text{ has } n_2, m_2 \text{ vertices and edges.}$ It is easy to see that $G_1 \times G_2$ is not isomorphic to $G_2 \times G_1$.

\[ G_1 = K_2 = \begin{array}{c}
\text{u}_1 \quad \text{v}_1 \\
\text{u}_2 \quad \text{v}_2 \end{array} \quad \text{and} \quad G_2 = K_{1,2} = \begin{array}{c}
\text{u}_1 \quad \text{v}_1 \\
\text{u}_2 \quad \text{v}_2 \end{array} \]

\[ G_1 \times G_2: \]

\[ (u_1, v_1) \quad (u_2, v_1) \quad (u_1, v_2) \quad (u_2, v_2) \]

3. The Cordiality of The Product of a Path and a Cycle

In this section, we show that the Cartesian product graphs $P_n \times C_m$ is cordial for some $n$.

**Lemma 3.1**: The graph $P_n \times C_m$ is cordial if $n$ is an even number.

**Proof**: We shall study two cases for $n \equiv i(mod \ 4)$ and $i = 0, 2$. Suppose that $n = 2s$

Case (1): At $i = 0$ we choose the following labeling:
The $[1^{st}, 3^{rd}, 5^{th} \ldots, (2s - 3)^{th}]$ rows are labeled by $O_{2s}$.
The $[2^{nd}, 4^{th}, 6^{th} \ldots, (2s - 2)^{th}]$ rows are labeled by $1_{2s}$.
The $(2s - 1)^{th}$ row is labeled by $M_{2s}$.
The $(2s)^{th}$ row is labeled by $L_{2s}$. In this case:

$V_0 = V_1 = [(2s)(s - 1)] + [s + s] = 2s^2$.

$E_0$ = the vertical edges plus the horizontal edges $= [s + s] + [2s(2s - 2) + (s)] = s(4s - 1)$

$E_1$ = the vertical edges plus the horizontal edges $= [(2s)(2s - 3) + s + (s)] + [2s + s] = s(4s - 1)$. Therefore $|E_0 - E_1| = 0$ and $|v_1 - v_1| = 0$ which satisfies the cordality conditions.

**Case (2):** For $i = 2$ we can apply the following labeling:

The $[1^{st}, 3^{rd}, 5^{th}, ..., (2s - 3)^{th}]$ rows are labeled by $O_{2s}$.

The $[2^{nd}, 4^{th}, 6^{th}, ..., (2s - 2)^{th}]$ rows are labeled by $O_{2s}$.

The $(2s - 1)^{th}$ row is labeled by $M_{2s}$.

The $(2s)^{th}$ row is labeled by $L_{2s - 10}$. In this situation:

$V_0 = V_1 = [(2s)(s - 1)] + [s + s] = 2s^2$.

$E_0$ = the horizontal edges plus the vertical edges $= [2s(2s - 2) + (s + 1)] + [(s - 1) + s] = s(4s - 1)$

$E_1$ = the horizontal edges plus the vertical edges $= [2s + (s - 1)] + [(2s)(2s - 3) + s + (s + 1)] = s(4s - 1)$. Therefore $|E_0 - E_1| = 0$, $|v_1 - v_1| = 0$ so $P_n \times C_n$ where $n$ is even is cordial as we wished to show. The following examples demonstrate some examples for the previous cases:

$$P_4 \times C_4 \ (for\ s = 2) \ v_0 = v_1 = 8, E_0 = E_1 = 14 \ i.e. |E_0 - E_1| = 0, \ |v_1 - v_1| = 0.$$  

![Figure 3.1](image1)

$$P_6 \times C_6 \ (for\ s = 3) \ v_0 = v_1 = 18, E_0 = E_1 = 33, i.e. |E_0 - E_1| = 0, \ |v_0 - v_1| = 0.$$  

![Figure 3.2](image2)

**Lemma 3.2:** The graph $P_n \times C_m$ is cordial for all $n$ is even and $m$ is odd such that $n - m = -1$ also it is cordial for all $n$ is odd and $m$ is even which satisfy $n - m = 1$.

**Proof:** We divide the proof into two cases:
Case (1): Suppose that \( n = 2s \) and \( m = 2s + 1 \) that satisfy the previous condition i.e. \( n - m = -1 \). Then we can use the following labeling:

The \([1^{st}, 3^{rd}, 5^{th}, \ldots, (2s - 3)^{th}]\) rows are labeled by \( O_{2s+1} \).

The \([2^{nd}, 4^{th}, 6^{th}, \ldots, (2s - 2)^{th}]\) rows are labeled by \( 1_{2s+1} \).

The \((2s - 1)^{th}\) row is labeled by \( M_{2s+1} \).

Then we will get: \( v_0 = v_1 = [(2s + 1)(s - 1)] + [s + 1 + s] = s(2s + 1) \),

\( E_0 \) = the vertical edges plus the horizontal edges \= [\( s + 2s \) + \[(2s + 1)(2s - 2) + (1 + 1)\] = \( s(4s + 1) \)

\( E_1 \) = the vertical edges plus the horizontal edges \= \[ s(2s - 3) + (2s - 1) + s(2s - 2) \] = \( s(4s + 1) - 1 \).

i.e. \( |E_0 - E_1| = 1, |v_0 - v_1| = 0 \) so \( P_{2s} \times C_{2s+1} \) is cordial as we wanted to show.

For example: \( (for \ s = 2) P_4 \times C_5 \)

\( v_0 = v_1 = 10, E_0 = 18, E_1 = 17, i.e. |E_0 - E_1| = 1, |v_0 - v_1| = 0 \)

![Figure 3.3](image1.png)

Case (2): Suppose that \( n = 2s + 1 \) and \( m = 2s \). It is obvious that \( n - m = 1 \).

By using the following labeling:

The \([1^{st}, 3^{rd}, 5^{th}, \ldots, (2s - 1)^{th}]\) rows are labeled by \( O_{2s} \).

The \([2^{nd}, 4^{th}, 6^{th}, \ldots, (2s)^{th}]\) rows are labeled by \( 1_{2s} \).

The \((2s + 1)^{th}\) row is labeled by \( M_{2s} \). Therefore,

\( v_0 = v_1 = [(2s)(s)] + [s] = s(2s + 1) \),

\( E_0 \) = the vertical edges plus the horizontal edges \= \[ s + [(2s)(2s)] \] = \( s(4s + 1) \).

\( E_1 \) = the vertical edges plus the horizontal edges \= \[ s(2s) + s(2s - 1) + [2s] \] = \( s(4s + 1) \).

i.e. \( |E_0 - E_1| = 1, |v_0 - v_1| = 0 \) so \( P_{2s+1} \times C_{2s} \) is cordial as we wanted to show.

For example: \( (for \ s = 2) P_5 \times C_4 \)

\( v_0 = v_1 = 10, E_0 = E_1 = 18, |E_0 - E_1| = 1, |v_0 - v_1| = 0 \).

![Figure 3.4](image2.png)
Lemma 3.3: The graph $P_n \times C_n$ is cordial if $n$ is an odd number.

Proof: Suppose that $n = 2s + 1$. Then one can take the following labeling:

The $[1^{st}, 3^{rd}, 5^{th}, \ldots, (2s - 1)^{th}]$ rows are labeled by $O_{2s+1}$.

The $[2^{nd}, 4^{th}, 6^{th}, \ldots, (2s)^{th}]$ rows are labeled by $1_{2s+1}$.

The $(2s + 1)^{th}$ row is labeled by $M_{2s+1}$. Therefore,

$v_0 = [(2s + 1)(s)] + [s + 1] = 2s(s + 1) + 1, v_1 = [(2s + 1)(s)] + [s] = 2s(s + 1)$.

$E_0 =$ the vertical edges plus the horizontal edges $= [s] + [2s (2s + 1) + 1] = s(4s + 3) + 1$.

$E_1 =$ the vertical edges plus the horizontal edges $= [(s + 1) + (2s + 1) (2s - 1)] + [2s] = s(4s + 3)$.

i.e. $|E_0 - E_1| = 1, |v_0 - v_1| = 1$ so $P_{2s+1} \times C_{2s+1}$ is cordial as we wanted to prove.

For example: (for $s = 2$) $P_5 \times C_5$

$v_0 = 13, v_1 = 12, E_0 = 23, E_1 = 22 \text{ i.e. } |E_0 - E_1| = 1, |v_0 - v_1| = 1$.

Now, from the previous lemmas we have the following conclusion:

Theorem 3.1: The Cartesian product graph $P_{2s+i} \times C_{2s+j}$ is cordial for all $i$ and $j$ where $(i, j = 0, 1)$.

4. The Cartesian Product of Two Paths

In this section, we study the cordiality of Cartesian product between two paths.

Lemma 4.1: The graph $P_n \times P_n$ is cordial if $n$ is an even number.

Proof: Suppose that $n = 2s$, then we choose the following labeling:

The $[1^{st}, 3^{rd}, 5^{th}, \ldots, (2s - 1)^{th}]$ rows are labeled by $O_{2s}$.

The $[2^{nd}, 4^{th}, 6^{th}, \ldots, (2s)^{th}]$ rows are labeled by $1_{2s}$. Then we get: $v_0 = v_1 = s(2s)$.

$E_0 =$ the vertical edges plus the horizontal edges $= 0 + [2s (2s - 1)] = 2s(2s - 1)$

$E_1 =$ the vertical edges plus the horizontal edges $= [2s (2s - 1)] + 0 = 2s(2s - 1)$

i.e. $|E_0 - E_1| = 0, |v_0 - v_1| = 0$. So, $P_{2s} \times P_{2s}$ is cordial as we wanted to show.

For example, $P_4 \times P_4$:

$v_0 = v_1 = 8, E_0 = E_1 = 12 \text{ i.e. } |E_0 - E_1| = 0, |v_0 - v_1| = 0$.
Lemma 4.2: The graph $P_n \times P_m$ is cordial for all $n$, where $n$ is even and $m$ is odd such that $n - m = -1$ also it is cordial for all $n$ is odd and $m$ is even such that $n - m = 1$.

Proof: We divide the proof into two cases:

Case (1): Suppose that $n = 2s$ and $m = 2s + 1$ which satisfy the previous condition $n - m = -1$. Therefore, we can use the following labeling:
The $[1^{st}, 3^{rd}, 5^{th}, ..., (2s - 1)^{th}]$ rows are labeled by $O_{2s+1}$.
The $[2^{nd}, 4^{th}, 6^{th}, ..., (2s)^{th}]$ rows are labeled by $1_{2s+1}$. In this case:

$v_0 = v_1 = s(2s + 1)$,
$E_0 =$ the vertical edges plus the horizontal edges $= [0] + [2s(2s)] = 4s^2$,
$E_1 =$ the vertical edges plus the horizontal edges $= [(2s + 1)(2s - 1)] + 0 = (2s + 1)(2s - 1)$,
i.e. $|E_0 - E_1| = 1$, $|v_0 - v_1| = 0$ so $P_{2s} \times P_{2s+1}$ is cordial as we wanted to show.

For example $P_4 \times P_5$:

Case (2): Suppose that $n = 2s + 1$ and $m = 2s$. It is obvious that $n - m = 1$. Therefore, we use the following labeling:
The $[1^{st}, 3^{rd}, 5^{th}, ..., (2s - 1)^{th}]$ columns are labeled by $O_{2s+1}$.
The $[2^{nd}, 4^{th}, 6^{th}, ..., (2s)^{th}]$ columns are labeled by $1_{2s+1}$. In this case:

$v_0 = v_1 = s(2s + 1)$,
$E_0 =$ the vertical edges plus the horizontal edges $= [2s(2s)] + 0 = (4s^2)$,
$E_1 =$ the vertical edges plus the horizontal edges $= 0 + [(2s + 1)(2s - 1)] = (2s + 1)(2s - 1)$, i.e. $|E_0 - E_1| = 1$, $|v_0 - v_1| = 0$ so $P_{2s+1} \times P_{2s}$ is cordial as we wanted to show.

For example $P_5 \times P_4$:

$v_0 = v_1 = 10$, $E_0 = 16$, $E_1 = 15$, i.e. $|E_0 - E_1| = 1$, $|v_0 - v_1| = 0$. 

Figure 4.1.

Figure 4.2.
**Lemma 4.3:** The graph $P_n \times P_n$ is cordial for all odd numbers $n$.

**Proof:** Suppose that $n = 2s + 1$. Therefore, one we take the following labeling:
The $[1^{st}, 2^{nd}]$ columns are labeled by the sequence $[1_s 0_{s+1}]$.
The $[3^{rd}, 5^{th}, ..., (2s + 1)^{th}]$ columns are labeled by the sequence $[0_s 1_{s+1}]$.
The $[4^{th}, 6^{th}, ..., (2s)^{th}]$ columns are labeled by the sequence $[1_s 0_{s+1}]$.

Then:
- $v_0 = (s + 1)(s + 1) + s(s) = 2s(s + 1) + 1$, $v_1 = s(s + 1) + s(s + 1) = 2s(2s + 1)$,
- $E_0 =$ the vertical edges plus the horizontal edges $= [(2s - 1)(s + 2) + (2s - 1)(s - 1)] + [(2s + 1)] = 2s(2s + 1)$,
- $E_1 =$ the vertical edges plus the horizontal edges $= [(2s + 1)] + [(2s - 1)(2s + 1)] = 2s(2s + 1)$, i.e. $|E_0 - E_1| = 0$, $|v_0 - v_1| = 1$. So $P_{2s+1} \times P_{2s+1}$ is cordial as we wanted to show.

For example $P_5 \times P_5$: $v_0 = 13$, $v_1 = 12$, $E_0 = E_1 = 20$, i.e. $|E_0 - E_1| = 0$, $|v_0 - v_1| = 1$.

The prove of the following theorem comes directly from the previous lemmas

**Theorem 4.1:** The Cartesian product graph $P_{2s+i} \times P_{2s+j}$ is cordial for all $i$ and $j$ where $(i, j = 0, 1)$.

**5. The Cartesian Product of a Cycle and A Path**

In this section, we will study the cordiality of a cycle and a path.

**Lemma 5.1:** The graph $C_n \times P_n$ is cordial if $n$ is an even number.

**Proof:** Suppose that $n = 2s$. Therefore, we study two cases for $n \equiv i(mod 4)$ and $i = 0, 2$.

**Case (1):** At $i = 0$ we choose the following labeling:
The $[1^{st}, 3^{rd}, 5^{th}, ..., (2s - 3)^{th}]$ columns are labeled by $0_{2s}$.
The $[2^{nd}, 4^{th}, 6^{th}, ..., (2s - 2)^{th}]$ columns are labeled by $1_{2s}$.

The $(2s - 1)^{th}$ column is labeled by $M_{2s}$. The $(2s)^{th}$ column is labeled by $L_{2s}$. Then:

$V_0 = V_1 = [(2s)(s - 1)] + [s + s] = 2s^2$.

$E_0 =$the vertical edges plus the horizontal edges $= [2s(2s - 2) + (s)] + [s + s] = s(4s - 1)$

$E_1 =$the vertical edges plus the horizontal edges $= [2s + s] + [(2s)(2s - 3) + s + (s)] = s(4s - 1)$ Then: $|E_0 - E_1| = 0, |v_0 - v_1| = 0$ which satisfies the cordiality conditions.

For example: $C_4 \times P_2$ for $s = 2$

$v_0 = v_1 = 8, E_0 = E_1 = 14 \text{ i.e: } |E_0 - E_1| = 0, |v_0 - v_1| = 0$.

![Figure 5.1](image)

**Case (2):** For $i = 2$ we can apply the following labeling:

The $[1^{st}, 3^{rd}, 5^{th}, ..., (2s - 3)^{th}]$ columns are labeled by $O_{2s}$.

The $[2^{nd}, 4^{th}, 6^{th}, ..., (2s - 2)^{th}]$ columns are labeled by $1_{2s}$.

The $(2s - 1)^{th}$ column is labeled by $M_{2s}$.

The $(2s)^{th}$ column is labeled by $L_{2s - 2}$. In this case:

$V_0 = V_1 = [(2s)(s - 1)] + [s + (s - 1) + 1] = 2s^2$.

$E_0 =$the vertical edges plus the horizontal edges $= [2s(2s - 2) + 0 + (s + 1)] + [(s - 1) + s] = s(4s - 1)$

$E_1 =$the vertical edges plus the horizontal edges $= [2s + (s - 1)] + [(2s)(2s - 3) + s + (s + 1)] = s(4s - 1)$. Then: $|E_0 - E_1| = 0, |v_0 - v_1| = 0$ which is cordial.

For example: $C_6 \times P_6$

$v_0 = v_1 = 18, E_0 = E_1 = 33, \text{ i.e. } |E_0 - E_1| = 0, |v_0 - v_1| = 0$.

![Figure 5.2](image)
Lemma 5.2: The graph $C_n \times P_m$ is cordial for all $n$ is even and $m$ is odd such that $n - m = -1$ also it is cordial for all $n$ is odd and $m$ is even which satisfy $n - m = 1$.

Proof: we divide the proof into two cases:

Case (1): Suppose that $n = 2s$ and $m = 2s + 1$ which satisfy the previous condition $n - m = -1$. we can use the following labeling:

The $[1^{st}, 3^{rd}, 5^{th}, ..., (2s - 1)^{th}]$ columns are labeled by $O_{2s}$.
The $[2^{nd}, 4^{th}, 6^{th}, ..., (2s)^{th}]$ columns are labeled by $1_{2s}$.
The $(2s + 1)^{th}$ column is labeled by $M_{2s}$. In this case:

$V_0 = V_1 = (s)(2s) + s = s(2s + 1)$,

$E_0$ = the vertical edges plus the horizontal edges $= [2s(2s) + (s)] = s(4s + 1)$

$E_1$ = the vertical edges plus the horizontal edges $= [2s] + [(2s)(s) + (2s - 1)(s)] = s(4s + 1)$

Then: $|E_0 - E_1| = 0, |v_0 - v_1| = 0$. So $C_{2s} \times P_{2s+1}$ is cordial as we wished.

For example: $C_4 \times P_5$

$v_0 = v_1 = 10, E_0 = E_1 = 18$ i.e. $|E_0 - E_1| = 0, |v_0 - v_1| = 0$.

![Figure 5.3]

Case (2): Suppose that $n = 2s + 1$ and $m = 2s$. It is obvious that $n - m = 1$.

We use the following labeling:

The $[1^{st}, 3^{rd}, 5^{th}, ..., (2s - 3)^{th}]$ columns are labeled by $O_{2s+1}$.

The $[2^{nd}, 4^{th}, 6^{th}, ..., (2s - 2)^{th}]$ columns are labeled by $1_{2s+1}$.

The $(2s - 1)^{th}$ column is labeled by $M_{2s+1}$.

The $(2s)^{th}$ column is labeled by $M_{2s+1}$. In this situation:

$v_0 = v_1 = (2s + 1)(s - 1) + s + (s + 1) = s(2s + 1)$,

$E_0$ = the vertical edges plus the horizontal edges $= [(2s + 1)(2s - 2) + 1 + 1] + [0 + s + 2s] = s(4s + 1)$,

$E_1$ = the vertical edges plus the horizontal edges $= [2s + 2s] + [s(2s - 3) + s(2s - 2) + (2s - 1)] = s(4s + 1) - 1$. Then $|E_0 - E_1| = 1, |v_0 - v_1| = 0$ which satisfy cordiality conditions.

For example: $C_5 \times P_4$

$v_0 = v_1 = 10, E_0 = E_1 = 17$ then $|E_0 - E_1| = 0, |v_0 - v_1| = 0$. 

---

Lemma 5.3. The graph \( C_n \times P_n \) is cordial for all \( n \) is an odd number.

**Proof:** Suppose that \( n = 2s + 1 \) we take the following labeling:
The \([1^{st}, 3^{rd}, 5^{th}, ..., (2s-1)^{th}]\) columns are labeled by \( O_{2s+1} \).
The \([2^{nd}, 4^{th}, 6^{th}, ..., (2s)^{th}]\) columns are labeled by \( 1_{2s+1} \).
The \((2s + 1)^{th}\) column is labeled by \( M_{2s+1} \).

Then:\( v_0 = (2s + 1)(s) + (s + 1) = 2s(s + 1) + 1 \), \( v_1 = (2s + 1)(s) + s = 2s(s + 1) \), \( E_0 \) = the vertical edges plus the horizontal edges = \([2s + 1(2s) + 1] + [s] = s(4s + 3) + 1\), \( E_1 \) = the vertical edges plus the horizontal edges = \([2s] + [(2s - 1)(2s + 1) + (s + 1)] = s(4s + 3)\) then \(|E_0 - E_1| = 1, |v_1 - v_1| = 1\). So \( C_{2s+1} \times P_{2s+1} \) is cordial as we wanted to show.

For example: \( C_5 \times P_5 \)
\( v_0 = 13, v_1 = 12, E_0 = 23, E_1 = 22 \) then \(|E_0 - E_1| = 1, |v_0 - v_1| = 1\).

As a direct consequence of the last three lemmas we have

**Theorem 5.1:** The Cartesian product graph \( C_{2s+i} \times P_{2s+j} \) is cordial for all \( i, j \) where \( (i, j = 0, 1) \).

**References**


*Corresponding author.  
E-mail address: mal-shamiri@kku.edu.sa