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ON THE POSITIVE PELL EQUATION $Y^2 = 72X^2 + 36$

M.A.Gopalan [1], A.Kavitha [2], A.Jesintha Mary [3]

[1] Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, India

[2] Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, India

[3] M.Phil Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, India

Abstract

The binary quadratic equation represented by the positive pellian $y^2 = 72x^2 + 36$ is analysed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

Keywords: Binary Quadratic; Hyperbola; Parabola; Integral Solutions; Pell Equation.

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1. Introduction

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 72x^2 + 36$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

2. Method of Analysis

Consider the binary quadratic equation

$$y^2 = 72x^2 + 36 \tag{1}$$

whose smallest positive integer solution is $x_0 = 2, y_0 = 18$

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To obtain the other solutions of (1), consider the pell equation $y^2 = 72x^2 + 1$ whose solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{72}} g_n, \tilde{y}_n = \frac{1}{2} f_n \tag{2}$$

where,

$$f_n = \left(17 + 2\sqrt{72}\right)^{n+1} + \left(17 - 2\sqrt{72}\right)^{n+1}$$

$$g_n = \left(17 + 2\sqrt{72}\right)^{n+1} - \left(17 - 2\sqrt{72}\right)^{n+1}, n = 0,1,2,3,....$$

Applying Brahamagupta Lemma between $\left(x_0, y_0\right)$and$\left(\tilde{x}_n, \tilde{y}_n\right)$, the other integer solutions of (1) are given by

$$\sqrt{72} x_{n+1} = \sqrt{72} f_n + 9 g_n$$

$$y_{n+1} = 9 f_n + \sqrt{72} g_n$$

The recurrence relations satisfied by the solutions (2) are given by

$$x_{n+3} - 34x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 34y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x & y satisfying (1) are given in the Table 1 below:

Table 1: Examples

n	x_n	y_n
0	2	18
1	70	594
2	2378	20178
3	80782	685458
4	2744210	23285394

From the above table, we observe some interesting relations among the solutions which are presented below:

1) x_n and y_n values are always even.
2) Each of the following expressions is a nasty number:

- $3x_{2n+3} - 99x_{2n+2} + 12$
- $\dfrac{3x_{2n+4} - 3363x_{2n+2} + 408}{34}$
- $\dfrac{6y_{2n+3} - 1680x_{2n+2} + 204}{17}$
- $\dfrac{6y_{2n+4} - 57072x_{2n+2} + 6924}{577}$
- $\dfrac{198y_{2n+2} - 48x_{2n+3} + 204}{17}$
- $198y_{2n+3} - 1680x_{2n+3} + 12$

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- $\frac{198y_{2n+4} - 57072x_{2n+3} + 204}{17}$
- $\frac{6726y_{2n+2} - 48x_{2n+4} + 6924}{577}$
- $\frac{6726y_{2n+3} - 1680x_{2n+4} + 204}{17}$
- $6726y_{2n+4} - 57072x_{2n+4} + 12$
- $99x_{2n+4} - 3363x_{2n+3} + 12$
- $\frac{35y_{2n+2} - y_{2n+3} + 36}{3}$
- $\frac{1189y_{2n+2} - y_{2n+4} + 1224}{102}$
- $\frac{1189y_{2n+3} - 35y_{2n+4} + 36}{3}$

3) Each of the following expressions is a cubical integer:

- $4[x_{3n+4} - 33x_{3n+3} + 12(x_{n+2} - 33x_{n+1})]$
- $4624[x_{3n+5} - 1121x_{3n+3} + 3(x_{n+3} - 1121x_{n+1})]$
- $23409[9y_{3n+4} - 2520x_{3n+3} + 3(9y_{n+2} - 2520x_{n+1})]$
- $26967249[9y_{3n+5} - 85608x_{3n+3} + 3(9y_{n+3} - 85608x_{n+1})]$
- $23409[297y_{3n+3} - 72x_{3n+4} + 3(297y_{n+1} - 72x_{n+2})]$
- $3[297y_{3n+4} - 2520x_{3n+4} + 3(297y_{n+2} - 2520x_{n+2})]$
- $23409[297y_{3n+5} - 85608x_{3n+4} + 3(297y_{n+3} - 85608x_{n+2})]$
- $26967249[10089y_{3n+3} - 72x_{3n+5} + 3(10089y_{n+1} - 72x_{n+3})]$
- $23409[10089y_{3n+4} - 2520x_{3n+5} + 3(10089y_{n+2} - 2520x_{n+3})]$
- $3[10089y_{3n+5} - 85608x_{3n+5} + 3(10089y_{n+3} - 85608x_{n+3})]$
- $324[297x_{3n+5} - 10089x_{3n+4} + 3(297x_{n+3} - 10089x_{n+2})]$
- $324[35y_{3n+3} - y_{3n+4} + 3(35y_{n+1} - y_{n+2})]$
- $374544[1189y_{3n+3} - y_{3n+5} + 3(1189y_{n+1} - y_{n+3})]$
- $324[1189y_{3n+4} - 35y_{3n+5} + 3(1189y_{n+2} - 35y_{n+3})]$

4) Relations among the solutions:

- $18x_{n+3} = 612x_{n+2} - 18x_{n+1}$
- $18y_{n+1} = 9x_{n+2} - 153x_{n+1}$
- $18y_{n+2} = 153x_{n+2} - 9x_{n+1}$
- $18y_{n+3} = 5193x_{n+2} - 153x_{n+1}$
- $612y_{n+1} = 9x_{n+3} - 5193x_{n+1}$

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- $612y_{n+2} = 153x_{n+3} - 153x_{n+1}$
- $612y_{n+3} = 5193x_{n+3} - 9x_{n+1}$
- $153y_{n+1} = 9y_{n+2} - 1296x_{n+1}$
- $5193y_{n+1} = 9y_{n+3} - 44064x_{n+1}$
- $5193y_{n+2} = 153y_{n+3} - 1296x_{n+1}$
- $9y_{n+1} = 153y_{n+2} - 1296x_{n+2}$
- $153y_{n+1} = 153y_{n+3} - 44064x_{n+2}$
- $153y_{n+2} = 9y_{n+3} - 1296x_{n+2}$
- $153y_{n+3} = 9y_{n+2} + 1296x_{n+3}$
- $9x_{n+2} = -18y_{n+3} + 153x_{n+3}$
- $9y_{n+1} = 5193y_{n+3} - 44064x_{n+3}$
- $18y_{n+1} = 153x_{n+3} - 5193x_{n+2}$
- $18y_{n+2} = 9x_{n+3} - 153x_{n+2}$
- $1296x_{n+3} = -153y_{n+1} + 5193y_{n+2}$
- $18y_{n+3} = -18y_{n+1} + 612y_{n+2}$

3. Remarkable Observation

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table 2 below:

Table 2: Hyperbolas

S.no	(X, Y)	Hyperbola
1	$(35x_{n+1} - x_{n+2}, x_{n+2} - 33x_{n+1})$	$81Y^2 - 72X^2 = 1296$
2	$(1189x_{n+1} - x_{n+3}, x_{n+3} - 1121x_{n+1})$	$81Y^2 - 72X^2 = 1498176$
3	$(297x_{n+1} - y_{n+2}, 9y_{n+2} - 2520x_{n+1})$	$Y^2 - 72X^2 = 93636$
4	$(10089x_{n+1} - y_{n+3}, 9y_{n+3} - 85608x_{n+1})$	$Y^2 - 72X^2 = 107868996$
5	$(9x_{n+2} - 35y_{n+1}, 297y_{n+1} - 72x_{n+2})$	$Y^2 - 72X^2 = 93636$
6	$(297x_{n+2} - 35y_{n+2}, 297y_{n+2} - 2520x_{n+2})$	$Y^2 - 72X^2 = 324$
7	$(10089x_{n+2} - 35y_{n+3}, 297y_{n+3} - 85608x_{n+2})$	$Y^2 - 72X^2 = 93636$
8	$(9x_{n+3} - 1189y_{n+1}, 10089y_{n+1} - 72x_{n+3})$	$Y^2 - 72X^2 = 107868996$
9	$(297x_{n+3} - 1189y_{n+2}, 10089y_{n+2} - 2520x_{n+3})$	$Y^2 - 72X^2 = 93636$
10	$(10089x_{n+3} - 1189y_{n+3}, 10089y_{n+3} - 85608x_{n+3})$	$Y^2 - 72X^2 = 324$
11	$(1189x_{n+2} - 35x_{n+3}, 297x_{n+3} - 10089x_{n+2})$	$Y^2 - 72X^2 = 1296$

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12	$\left(y_{n+2}-33y_{n+1}, 35y_{n+1}-y_{n+2}\right)$	$288Y^2-324X^2=373248$
13	$\left(y_{n+3}-1121y_{n+1}, 1189y_{n+1}-y_{n+3}\right)$	$2312Y^2-2601X^2=3463782912$
14	$\left(297y_{n+3}-10089y_{n+2}, 1189y_{n+2}-35y_{n+3}\right)$	$72Y^2-X^2=93312$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table 3 below:

Table 3: Parabolas

S.no	(X,Y)	Parabola
1	$\left(35x_{n+1}-x_{n+2}, x_{2n+3}-33x_{2n+2}\right)$	$72X^2=162Y-648$
2	$\left(1189x_{n+1}-x_{n+3}, x_{2n+4}-1121x_{2n+2}\right)$	$72X^2=5508Y-749088$
3	$\left(297x_{n+1}-y_{n+2}, 9y_{2n+3}-2520x_{2n+2}\right)$	$72X^2=153Y-46818$
4	$\left(10089x_{n+1}-y_{n+3}, 9y_{2n+4}-85608x_{2n+2}\right)$	$72X^2=5193Y-53934498$
5	$\left(9x_{n+2}-35y_{n+1}, 297y_{2n+2}-72x_{2n+3}\right)$	$72X^2=153Y-46818$
6	$\left(297x_{n+2}-35y_{n+2}, 297y_{2n+3}-2520x_{2n+3}\right)$	$72X^2=9Y-162$
7	$\left(10089x_{n+2}-35y_{n+3}, 297y_{2n+4}-85608x_{2n+3}\right)$	$72X^2=153Y-46818$
8	$\left(9x_{n+3}-1189y_{n+1}, 10089y_{2n+2}-72x_{2n+4}\right)$	$72X^2=5193Y-539344$
9	$\left(297x_{n+3}-1189y_{n+2}, 10089y_{2n+3}-2520x_{2n+4}\right)$	$72X^2=153Y-46818$
10	$\left(10089x_{n+3}-1189y_{n+3}, 10089y_{2n+4}-85608x_{2n+4}\right)$	$72X^2=9Y-162$
11	$\left(1189x_{n+2}-35x_{n+3}, 297x_{2n+4}-10089x_{2n+3}\right)$	$72X^2=18Y-648$
12	$\left(y_{n+2}-33y_{n+1}, 35y_{2n+2}-y_{2n+3}\right)$	$X^2=16Y-576$
13	$\left(y_{n+3}-1121y_{n+1}, 1189y_{2n+2}-y_{2n+4}\right)$	$X^2=544Y-665856$
14	$\left(297y_{n+3}-10089y_{n+2}, 1189y_{2n+3}-35y_{2n+4}\right)$	$X^2=1296Y-46656$

III. Consider $m=x_{n+1}+y_{n+1}, n=x_{n+1}$. Observe that $m>n>0$. Treat m, n as the generators of the Pythagorean Triangle $T(\alpha,\beta,\gamma)$, where $\alpha=2mn$, $\beta=m^2-n^2$, $\gamma=m^2+n^2$

Then the following interesting relations are observed:

a) $\alpha-36\beta+35\gamma=-36$

b) $37\alpha-\gamma+\dfrac{144A}{P}=36$

c) $-34\gamma+36\beta-38\alpha+\dfrac{144A}{P}=72$

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d) $\frac{2A}{P} = x_{n+1}y_{n+1}$

4. Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive Pell equation $y^2 = 72x^2 + 36$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive Pell equations and determine their integer solutions along with suitable properties.

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