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TECHNIQUES TO SOLVE UNIFORM THIRD DEGREE EQUATION HAVING FOUR VARIABLES

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ABSTRACT

The uniform third degree equation having four variables given by $x^3 + y^3 = 7(z-w)^2(z+w)$ is studied to obtain its non-zero distinct integral solutions. Substitution technique and factorization method are utilized to determine the same.

Keywords: Techniques, Cubic Equation, Homogeneous, Unknowns

1. INTRODUCTION

The technological significance of uniform polynomial equations of degree three having four variables with coefficients in integers is wonderful as it is deeply related to number of problems in the theory of numbers. Particularly, the uniform cubic equation with four variables takes up a very important place that has a significant impact on the development of number theory in the realm of mathematics. It is wellknown that uniform or non-uniform cubic polynomial equations have attracted many mathematicians. For example refer Diophantine Equations, Vidhyalakshmi & Gopalan. (2022) for third degree equations. It is observed in Thiruniraiselvi & Gopalan (2021) the solutions presented by authors are erroneous. The above problem [16] motivated us to search for varieties of solutions in integers to uniform third degree equation having four variables presented in Vidhyalakshmi & Gopalan (2022) through the substitution technique and factorization method.

1.1. METHOD OF ANALYSIS

The homogeneous cubic equation with four unknowns to be solved is

$$x^3 + y^3 = 7(z-w)^2(z+w) \tag{1}$$

By inspection, it is seen that (1) is satisfied by the quadruples given by

However, there are many more choices of integer solutions to (1) and the process of obtaining the same is illustrated below:

Process 1.1.1

The substitution of the linear transformations

$$x = u+v,\ y = u-v,\ z = u+p,\ w = u-p\ ,u \neq v,\ p \tag{2}$$

in (1) leads to the homogeneous ternary quadratic equation

$$u^2 + 3v^2 = 28p^2 \tag{3}$$

Assume

$$p = a^2 + 3b^2 \tag{4}$$

Express the integer 28 on the R.H.S. of (3) as the product of complex conjugate as shown below:

$$28 = (5+i\sqrt{3})\,(5-i\sqrt{3}) \tag{5}$$

Substituting (4) & (5) in (3) and utilizing factorization, we consider

$$u + i\sqrt{3}\,v = (5+i\sqrt{3})\,(a+i\sqrt{3}\,b)^2 \tag{6}$$

Equating the real and imaginary parts in (6), we get

$$u = 5(a^2-3b^2) - 6ab,$$
$$v = (a^2-3b^2) + 10ab. \tag{7}$$

In view of (2), one obtains the integer solutions to (1) to be

$$x = 6\,(a^2 - 3b^2) + 4ab,$$
$$y = 4\,(a^2 - 3b^2) - 16ab,$$
$$z = 6\,(a^2 - 2b^2) - 6ab,$$
$$w = 2\,(2a^2 - 9b^2) - 6ab\,. \qquad (8)$$

Note 1

In addition to (5), the integer 28 is written as

$$28 = (4 + i2\sqrt{3})\,(4 - i2\sqrt{3}).$$

Following the above procedure, a different pattern of integer solutions to (1) is obtained.

Process 1.1.2

Write (3) as

$$u^2 + 3v^2 = 28p^2 * 1 \qquad (9)$$

Assume the integer 1 on the R.H.S. of (9) as

$$1 = \frac{(1 + i\sqrt{3})\,(1 - i\sqrt{3})}{4} \qquad (10)$$

Substituting (4), (5) & (10) in (9) and employing factorization, we consider

$$u + i\sqrt{3}\,v = \frac{(5 + i\sqrt{3})(a + i\sqrt{3}\,b)^2\,(1 + i\sqrt{3})}{2} \qquad (11)$$

Equating the real and imaginary parts in (11), we have

$$u = (a^2 - 3b^2) - 18ab,$$
$$v = 3\,(a^2 - 3b^2) + 2ab$$

In view of (2), one obtains the integer solutions to (1) to be

$$x = 4\,(a^2 - 3b^2) - 16ab,$$
$$y = -2\,(a^2 - 3b^2) - 20ab,$$
$$z = 2a^2 - 18ab,$$
$$w = -6b^2 - 18ab\,. \qquad (12)$$

Note 2

In addition to (10), the integer 1 is written as

$$1 = \frac{(3r^2 - s^2 + i\sqrt{3}\,2rs)\,(3r^2 - s^2 - i\sqrt{3}\,2rs)}{(3r^2 + s^2)^2},$$

$$1 = \frac{(r^2 - 3s^2 + i\sqrt{3}\,2rs)\,(r^2 - 3s^2 - i\sqrt{3}\,2rs)}{(r^2 + 3s^2)^2}$$

Following the above procedure, one obtains varieties of solutions in integers for (1).

Process 1.1.3

The ratio form of (3) is

$$\frac{u+5p}{p+v}=\frac{3(p-v)}{u-5p}=\frac{\alpha}{\beta},\beta\neq 0$$

Solving the above system of double equations, we get

$$u=5\alpha^2+6\alpha\beta-15\beta^2,$$
$$v=-\alpha^2+10\,\alpha\beta+3\beta^2$$

and

$$p=\alpha^2+3\beta^2 \qquad (13)$$

From (2), observe that (1) is satisfied by

$$x=4\alpha^2+16\alpha\beta-12\beta^2,$$
$$y=6\alpha^2-4\alpha\beta-18\beta^2,$$
$$z=6\alpha^2+6\alpha\beta-12\beta^2,$$
$$w=4\alpha^2+6\alpha\beta-18\beta^2 \qquad (14)$$

jointly with (13).

Note 3

One may also express (3) in the ratio forms as below:

$$\frac{u+5p}{p-v}=\frac{3(p+v)}{u-5p}=\frac{\alpha}{\beta},\beta\neq 0,$$
$$\frac{u+5p}{3(p-v)}=\frac{(p+v)}{u-5p}=\frac{\alpha}{\beta},\beta\neq 0,$$
$$\frac{u+5p}{3(p+v)}=\frac{(p-v)}{u-5p}=\frac{\alpha}{\beta},\beta\neq 0.$$

The repetition of the above process gives three different choices of solutions in integers for (1).

Process 1.1.4

Rewrite (3) as

$$3v^2=28p^2-u^2 \qquad (15)$$

Assume

$$v=28a^2-b^2 \qquad (16)$$

Express 3 on the L.H.S. of (15) as

$$3=(\sqrt{28}+5)\ (\sqrt{28}-5) \tag{17}$$

Substituting (16) & (17) in (15) and applying factorization, we have

$$\sqrt{28}\,p+u=(\sqrt{28}+5)\ (\sqrt{28}\,a+b)^2$$

Equating the coefficients of corresponding terms, we get

$$u=5(28a^2+b^2)+56ab$$

and

$$p=28a^2+b^2+10ab \tag{18}$$

In view of (2), observe that (1) is satisfied by

$$x=168a^2+4b^2+56ab,$$
$$y=112a^2+6b^2+56ab,$$
$$z=6(28\ a^2+b^2)+66ab,$$
$$w=4(28\ a^2+b^2)+46ab \tag{19}$$

jointly with (18).

Note 4

In (17), 3 can be expressed as

$$3=\frac{(\sqrt{28}+4)(\sqrt{28}-4)}{4}$$

The repetition of the above process gives one more set of integer solutions to (1).

Process 1.1.5

Rewrite (3) as

$$28p^2-3v^2=1*u^2 \tag{20}$$

Assume

$$u=28a^2-3b^2 \tag{21}$$

Express 1 on the R.H.S. of (20) as

$$1=\frac{(\sqrt{28}+\sqrt{3})(\sqrt{28}-\sqrt{3})}{25} \tag{22}$$

Substituting (21), (22) & (17) in (20) and applying factorization, we have

$$\sqrt{28}p+\sqrt{3}v=\frac{(\sqrt{28}+\sqrt{3})}{5}(\sqrt{28}a+\sqrt{3}b)^2$$

Equating the coefficients of corresponding terms, we get

$$v=\frac{1}{5}(28a^2+3b^2+56ab)$$

and

$$p=\frac{1}{5}(28a^2+3b^2+6ab) \qquad (23)$$

Taking in the representations of u, v & p and using (2), it is seen that (1) is satisfied by

$$x=25(28A^2-3B^2)+5(28A^2+3B^2+56AB)$$
$$y=25(28A^2-3B^2)-5(28A^2+3B^2+56AB)$$
$$z=25(28A^2-3B^2)+5(28A^2+3B^2+6AB)$$
$$w=25(28A^2-3B^2)-5(28A^2+3B^2+6AB)$$
$$p=5(28A^2+3B^2+6AB) \qquad (24)$$

Note 5

In (22), 1 can be expressed as

$$1=\frac{(\sqrt{28}+2\sqrt{3})(\sqrt{28}-2\sqrt{3})}{16}$$

The repetition of the above process gives one more set of integer solutions to (1).

Pattern 1.1.6

In (1), the option

$$x=2u+2v,\ y=2u-2v,\ z=2u+p,\ w=2u-p \qquad (25)$$

reduces it to the homogeneous ternary quadratic equation

$$u^2+3v^2=7p^2 \qquad (26)$$

Express 7 on the R.H.S. of (26) to be

$$7=(2+i\sqrt{3})(2-i\sqrt{3}) \qquad (27)$$

Use (4) & (27) in (26). Utilizing factorization, we consider

$$u+i\sqrt{3}v=(2+i\sqrt{3})(a-3b+i2\sqrt{3}ab) \qquad (28)$$

On comparing the coefficients of corresponding terms in (6), one has

$$u = 2(a^2 - 3b^2) - 6ab$$

$$v = (a^2 - 3b^2) + 4ab \quad (29)$$

Applying (2), one obtains corresponding solutions in integers of (1).

2. CONCLUSION

This article concentrates on finding different choices of solutions in integers for uniform cubic Diophantine equation having four variables. One may attempt to find solutions in integers for various forms of cubic Diophantine equations having atleast four variables.

CONFLICT OF INTERESTS

None.

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