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ON THE LANZHOU INDEX OF GRAPHS

Qinghe Tong [1], Chengxu Yang [2], Wen Li [2]

[1] School of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai 810008, China
[2] School of Computer, Qinghai Normal University, Xining, Qinghai 810008, China

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Corresponding Author
Chengxu Yang, cxuyang@aliyun.com

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ABSTRACT

Let $G=(V(G),E(G))$ be a simple, connected, and nontrivial graph with vertex set $V(G)$ and edge set $E(G)$. The Lanzhou index of a graph G is defined by $\mathrm{Lz}(G)=\sum_{u\in V(G)} d_{\overline{G}}(v) d_G^2(v)$, where $d_G(v)$ is degree of the vertex v in G. In a chemical graph theory, the topological index can help determine chemical, biological, pharma-cological, toxicological, and technically relevant information on molecules. In this paper, we get exact formulas for $\mathrm{Lz}(G)$, where G is some certain chemical graphs, like silicate, chain silicate, oxide, and graphene networks. Moreover, we determine the Lanzhou index of several graph operations.

Keywords: Lanzhou Index, Silicate Network, Graph operation

1. INTRODUCTION

In this paper, we consider simple, connected, and finite graphs. For a vertex $v\in V(G)$, the number of all edge incidents with v is denoted by $d_G(v)$, the maximum (minimum) degree of G is denoted by $\Delta(G)(\delta(G))$.

Moreover, we let

$NV_i=\{v|d_G(v)=i,$ and $v\in V(G)\}$, $NE_{a,b}=\{uv|uv\in E(G), d_G(u)=a, d_G(v)=b\}$.

The topological index is very useful in chemistry, its value is related to the molecular structure-activity of the compound. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$, introduced by Kazemi in Kazemi and Behtoei (2017), are defined as

$$M_1(G) = \sum_{v \in V(G)} d_G^2(v) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)).$$
$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

Furtula and Gutman in Furtula and Gutman (2015) introduced the forgotten index of G, which is defined by

$$F(G) = \sum_{v \in V(G)} d_G^3(v) = \sum_{uv \in E(G)} (d_G^2(u) + d_G^2(v)).$$

After that, Vukičević et al. Vukičević et al. (2018). introduced an index named by Lanzhou index, which is defined as

$$\begin{aligned} \mathrm{Lz}(G) &= \sum_{v \in V(G)} d_{\bar{G}}(v)\, d_G^2(v) \\ &= \sum_{v \in V(G)} (n - 1 - d_G(u)) d_G^2(v) = (n-1)M_1(G) - F(G). \end{aligned}$$

2. RESULTS FOR LINE GRAPHS

Let $L(G) = (E(G), E_L)$ be the line graph of G. The vertex of corresponds to the edge of G. Two vertices of $L(G)$ are adjacent if and only if the corresponding edge of G is adjacent. Then we have $E_L = \{(u_1v, vu_2) | u_1v, vu_2 \in E(G)\}$. Line graphs are widely used in the field of chemistry. For example, Nadeem studied GA_5 and ABC_4 indices of the line graph of the wheel, tadpole in paper Nadeem et al. (2015). Randić found that the connectivity index of a line graph in some molecular graph is highly correlated with some physicochemical properties. Gutman further proposed the application of line graphs in physical chemistry. All this mean that in the field of topology, line graph is extremely important in the study of chemistry and physics. By definition of line graph, we have the following observation. From the definitions, the following observation is immediate.

2.1. RESULTS FOR LINE GRAPHS

Observation 1. Let G be a graph, for $v, u_1, u_2 \in V(G)$, and $u_1v, u_2v \in E(G)$, we have

$$d_{L(G)}(u_1v) = d_G(v) + d_G(u_1) - 2 \text{ and } d_{L(G)}(vu_2) = d_G(v) + d_G(u_2) - 2.$$

Theorem 2.1. Vukičević et al. (2018). Let G be a connected graph of order n. Then we have $\mathrm{Lz}(G) \geq 0$. The inequality is satisfied if and only if G is either complete or empty graph.

Theorem 2.2.

Let G be a connected graph with order $n \geq 5$ and $\delta \geq 2$. Then

$$0 \leq Lz(L(G)) \leq 2n(n-1)\Delta(G)(\Delta(G)-1)^2 - 4n\delta(G)(\delta(G)-1)^3.$$

With right equality holding if and only if and only if G is a regular graph and the lower bound is sharp.

Proof. For the up bound, by definition of Lanzhou index, we have

$$\begin{aligned}
\mathrm{Lz}(L(G)) &= \sum_{(u_1v,vu_2)\in E_L} ((n-1)M_1(L(G)) - F(L(G))) \\
&= \sum_{(u_1v,vu_2)\in E_L} \left((n-1)(d_{L(G)}(u_1v)+d_{L(G)}(vu_2)) - (d^2_{L(G)}(u_1v)+d^2_{L(G)}(vu_2))\right) \\
&= \sum_{(u_1v,vu_2)\in E_L} \left((n-1)(d_{L(G)}(u_1v)+d_{L(G)}(vu_2))\right) - \sum_{(u_1v,vu_2)\in E_L} \left(d^2_{L(G)}(u_1v)+d^2_{L(G)}(vu_2)\right) \\
&= \sum_{\substack{v\in V(G)\\ d_G(v)\geq 2}} \sum_{\substack{u_1u_2\in N(v)\\ u_1\neq u_2}} \left((n-1)((d_G(v)+d_G(u_1)-2)+(d_G(v)+d_G(u_2)-2)\right) \\
&\quad - \sum_{\substack{v\in V(G)\\ d_G(v)\geq 2}} \sum_{\substack{u_1u_2\in N(v)\\ u_1\neq u_2}} \left((d_G(v)+d_G(u_1)-2)^2+(d_G(v)+d_G(u_2)-2)^2\right) \\
&\leq n\binom{\Delta(G)}{2}(n-1)(4\Delta(G)-4) - n\binom{\delta(G)}{2}2(2\delta(G)-2)^2 \\
&= 2n(n-1)\Delta(G)(\Delta(G)-1)^2 - 4n\delta(G)(\delta(G)-1)^3.
\end{aligned}$$

With equality holds if and only if G is a regular graph.

In addition, for the lower bound, from Theorem 2, it follows that $\mathrm{Lz}(G) \geq 0$. Let G is $\frac{n}{2}K_2$ or $K_1 \cup \frac{n-1}{2}K_2$ or star graph $K_{1,n-1}$, it follows that $\mathrm{Lz}(L(G))$ is an empty graph or a complement graph, from theorem 2, we have$\mathrm{Lz}(L(G)) = 0$. So, the lower bound is sharp.

2.2. RESULTS FOR LINE GRAPH OF GRAPHENE SHEET

Graphene $G(m,n)$ is a 2-dimensional layer of pure carbon. Line graph of graphene$G(7,5)$ is shown below in Figure 1

Figure 1

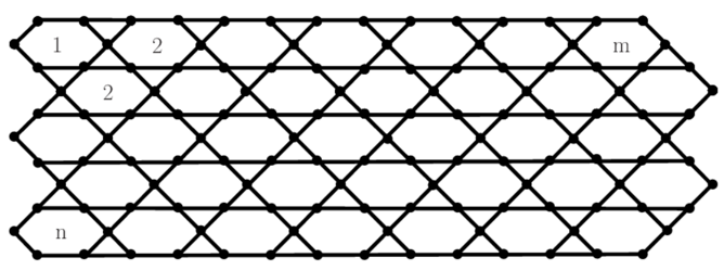

Figure 1 $\boldsymbol{L(G(7,5))}$

Theorem 2.3. Let n, m are two integers, $m \geq 1$, and let $G(m,n)$ be a molecular graph of graphene with n rows and m columns. Then

$$Lz(L(G(m,n))) = \begin{cases} 260m^2 - 312m + 124 & for \quad n=1; \\ 212 - 60m + 8m^2 - 396mn + 108m^2n + 144mn^2 & \\ -82n + 20mn + 114mn^2 + 12n^2 & for \quad n \geq 2. \end{cases}$$

Proof. For $n=1$, we have $|V(G(m,1))| = 5m+1$, $|NV_2| = 6$, $|NV_3| = 4m-4$, and $|NV_4| = m-1$. Let $G = L(G(m,1))$. From the definition of Lanzhou index, we have that

$$\begin{aligned} \mathrm{Lz}(G) &= \sum_{v\in V(G)} d_{\overline{G}}(v) d_G^2(v) \\ &= \sum_{v\in NV_2} d_{\overline{G}}(v) d_G^2(v) + \sum_{v\in NV_3} d_{\overline{G}}(v) d_G^2(v) + \sum_{v\in NV_4} d_{\overline{G}}(v) d_G^2(v) \\ &= \sum_{v\in NV_2} 2^2(5m-2) + \sum_{v\in NV_3} 3^2(5m-3) + \sum_{v\in NV_4} 4^2(5m-4) \\ &= 6(2^2(5m-2)) + (4m-4)(3^2(5m-3)) + (m-1)(4^2(5m-4)) \\ &= 260m^2 - 312m + 124. \end{aligned}$$

Similarly, for $n \geq 2$, we have $|V(G(m,n))| = 2n + 2m + 3mn - 1$, $|NV_2| = n+4$, $|NV_3| = 4m + 2n - 4$, and $|NV_4| = 3mn - 2m - n - 1$. Let $G = L(G(m,1))$. Hence

$$\begin{aligned} \mathrm{Lz}(G) &= \sum_{v\in V(G)} d_{\overline{G}}(v) d_G^2(v) \\ &= \sum_{v\in NV_2} d_{\overline{G}}(v) d_G^2(v) + \sum_{v\in NV_3} d_{\overline{G}}(v) d_G^2(v) + \sum_{v\in NV_4} d_{\overline{G}}(v) d_G^2(v) \\ &= \sum_{v\in NV_2} 2^2(2n+2m+3mn-4) + \sum_{v\in NV_3} 3^2(2n+2m+3mn-5) \\ &\quad + \sum_{v\in NV_4} 4^2(2n+2m+3mn-6) \\ &= (n+4)\left(2^2(2n+2m+3mn-4)\right) + (4m+2n-4)\left(3^2(2n+2m+3mn-5)\right) \\ &\quad + (3mn-2m-n-1)\left(4^2(2n+2m+3mn-6)\right) \\ &= 212 - 60m + 8m^2 - 396mn + 108m^2n + 144mn^2 - 82n + 20mn + 114mn^2 + 12n^2 \end{aligned}$$

2.3. RESULTS FOR LINE GRAPH OF DENDRIMER STARS

Nanostar dendrimers are generally synthesized by divergent or convergent methods. It is a kind of hyperbranched nanostructures. $NS_1[n]$ $NS_2[n]$ and $NS_3[n]$ are the line graph of three dendrimers and widely appear in drug structure, see Figure 2, Figure 3, Figure 4.

Figure 2

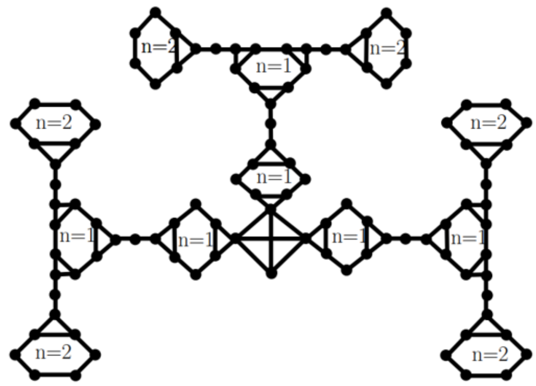

Figure 2 Line Graph of Nanostar Dendrimer $\boldsymbol{L(NS_1[2])}$

Figure 3

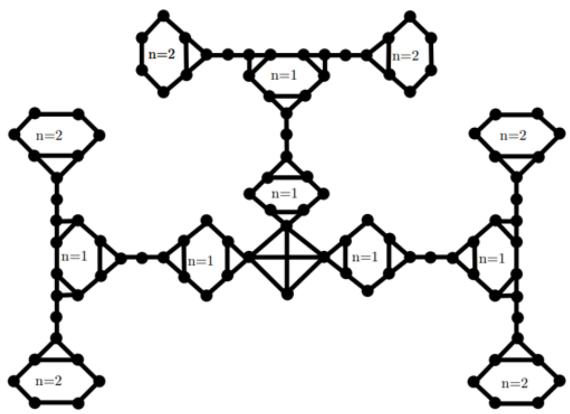

Figure 3 Line Graph of Nanostar Dendrimer $\boldsymbol{L(NS_2[2])}$

Figure 4

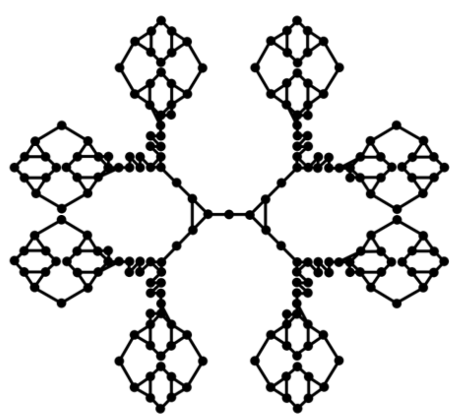

Figure 4 Line Graph of Nanostar Dendrimer $\boldsymbol{L(NS_3[2])}$

Theorem 2.4. Let n, m are two integers. Then the Lanzhou index of line graphs of three infinite classes $NS_1[n]$, $NS_2[n]$ and $NS_3[n]$ of dendrimer stars are

$$Lz(L(G(m,n)))\begin{cases} 2673\cdot 2^{2n+1}+1035\cdot 2^{n+1}-30, & for \quad i=1; \\ 297\cdot 2^{2n+5}-507\cdot 2^{n+3}+424, & for \quad i=2; \\ 3219\cdot 2^{2n+3}-769\cdot 2^{n+4}+1366, & for \quad i=3. \end{cases}$$

Where n is the number of steps of growth of these three families of dendrimer stars.

Proof. For $L(NS_1[n])$, we have $|V(L(NS_1[N]))| = 27\cdot 2^n - 5$, $|NV_2| = 9(2^n) + 3$, $|NV_3| = 18(2^n) - 11$, and $|NV_5| = 3$. Let $G = L(NS_1[n])$. We have

$$\begin{aligned} \mathrm{Lz}(G) &= \sum_{v\in V(G)} d_{\overline{G}}(v)d_G^2(v) \\ &= \sum_{v\in NV_2} d_{\overline{G}}(v)d_G^2(v) + \sum_{v\in NV_3} d_{\overline{G}}(v)d_G^2(v) + \sum_{v\in NV_5} d_{\overline{G}}(v)d_G^2(v) \\ &= \sum_{v\in NV_2} 2^2(27\cdot 2^n - 8) + \sum_{v\in NV_3} 3^2(27\cdot 2^n - 9) + \sum_{v\in NV_4} 5^2(27\cdot 2^n - 11) \\ &= (9\cdot 2^n + 3)\left(2^2(27\cdot 2^n - 8)\right) + (18\cdot 2^n - 11)\left(3^2(27\cdot 2^n - 9)\right) \\ &\quad +3\left(5^2(27\cdot 2^n - 11)\right) \\ &= 2673\cdot 2^{2n+1} + 1035\cdot 2^{n+1} - 30. \end{aligned}$$

Similarly, for $L(NS_2[n])$, we have $|V(L(NS_2[N]))| = 36\cdot 2^n - 5$, $|NV_2| = 12(2n-1) + 14$, $|NV_3| = 24(2n-1) + 16$ and $|NV_4| = 1$. Let $G = L(NS_2[n])$. Then we have

$$\begin{aligned} \mathrm{Lz}(G) &= \sum_{v\in V(G)} d_{\overline{G}}(v)d_G^2(v) = \sum_{v\in NV_2} d_{\overline{G}}(v)d_G^2(v) + \sum_{v\in NV_3} d_{\overline{G}}(v)d_G^2(v) + \sum_{v\in NV_4} d_{\overline{G}}(v)d_G^2(v) \\ &= \sum_{v\in NV_2} 2^2(36\cdot 2^n - 8) + \sum_{v\in NV_3} 3^2(36\cdot 2^n - 9) + \sum_{v\in NV_4} 4^2(36\cdot 2^n - 10) \\ &= \left(12(2^n - 1) + 14\right)\left(2^2(36\cdot 2^n - 8)\right) + \left(24(2^n - 1) + 16\right)\left(3^2(36\cdot 2^n - 9)\right) + 4^2(36\cdot 2^n - 10) \\ &= 297\cdot 2^{2n+5} - 507\cdot 2^{n+3} + 424. \end{aligned}$$

For $L(NS_3[n])$, we have $|V(L(NS_2[N]))| = 116\cdot 2^{n-1} - 13$, $|NV_2| = 48(2^{n-1} - 1) + 41$, $|NV_3| = 56(2^{n-1} - 1) + 50$, and $|NV_4| = 12(2^{n-1})$. Let $G = L(NS_3[n])$. By the definition of Lanzhou index, we have

$$\begin{aligned} \mathrm{Lz}(G) &= \sum_{v\in V(G)} d_{\overline{G}}(v)d_G^2(v) = \sum_{v\in NV_2} d_{\overline{G}}(v)d_G^2(v) + \sum_{v\in NV_3} d_{\overline{G}}(v)d_G^2(v) + \sum_{v\in NV_4} d_{\overline{G}}(v)d_G^2(v) \\ &= \sum_{v\in NV_2} 2^2(116\cdot 2^{n-1} - 16) + \sum_{v\in NV_3} 3^2(116\cdot 2^{n-1} - 17) + \sum_{v\in NV_4} 4^2(116\cdot 2^{n-1} - 18) \\ &= \left(48(2^{n-1} - 1) + 41\right)\left(2^2(116\cdot 2^{n-1} - 16)\right) + \left(56(2^{n-1} - 1) + 50\right)\left(3^2(116\cdot 2^{n-1} - 17)\right) \\ &\quad +12(2^{n-1})\left(4^2(116\cdot 2^{n-1} - 18)\right) \\ &= 3219\cdot 2^{2n+3} - 769\cdot 2^{n+4} + 1366. \end{aligned}$$

3. RESULTS FOR CHAIN SILICATE NETWORKS GRAPH

In this section, we consider a family of chain silicate networks. This network is denoted by CS_n and is obtained by arranging n tetrahedral linearly, see Figure 5.

Figure 5

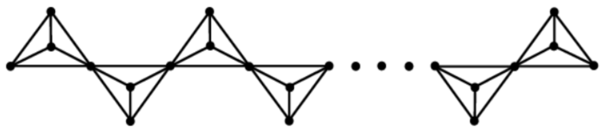

Figure 5 CS_n

We refer the reader to article Kazemi and Behtoei (2017) for some aspects of the parameters of silicate networks. Obviously, for any silicate networks CS_n, $|V(CS_n)| = 3n + 1$ and $|E(CS_n)| = 6n$.

Theorem 3.1. Let n be a positive integer, for an n-long silicate networks CS_n, we have $\text{Lz}(CS_n) = 162(n-1)^2$.

Proof. Obviously, $|V(CS_n)| = 3n + 1$. Note that CS_n is consist of n tetrahedrons which connected by linear chains. Then we have $|NV_3| = 2n + 2$ and $|NV_6| = n - 1$. Let $G = CS_n$,

we get

$$\begin{aligned}\text{Lz}(G) &= \sum_{v\in V(G)} d_{\overline{G}}(v) d_G^2(v) = \sum_{v\in NV_3} d_{\overline{G}}(v) d_G^2(v) + \sum_{v\in NV_6} d_{\overline{G}}(v) d_G^2(v) \\ &= \sum_{v\in NV_3} 3^2(3n-3) + \sum_{v\in NV_6} 6^2(3n-6) = (2n+2)\left(3^2(3n-3)\right) + (n-1)\left(6^2(3n-6)\right) \\ &= 162(n-1)^2.\end{aligned}$$

Define the honeycomb network as $HC(n)$. n is number of layers from the center to the boundary in $HC(n)$. We use a honeycomb network to constructe the silicate network $SL(n)$ by placing silicon ions on all the vertices of $HC(n)$, and dividing the edges and placing oxygenions on the new vertices, last placing oxygenions at the pendent vertices, where the silicate network defined as $SL(n)$. When $n = 2$, the silicate network is as follow.

Figure 6

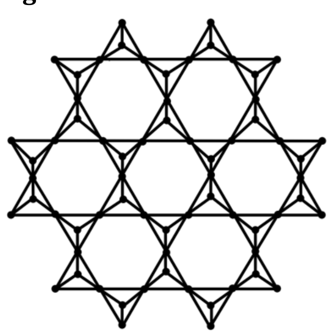

Figure 6 $SL(2)$

Obviously, for the silicate networks $SL(n)$, we have $|V(SL(n))| = 3n(5n + 1)$ and $|E(SL(n))| = 36n^2$. The set of edges can decompose as three types:

Type (1): $E^1(SL(n)) = \{uv | d_{SL(n)}(u) = 3 \ \ and \ \ d_{SL(n)}(v) = 3\}$

Type (2): $E^1(SL(n)) = \{uv | d_{SL(n)}(u) = 3 \ \ and \ \ d_{SL(n)}(v) = 6\}$

Type (3): $E^1(SL(n)) = \{uv | d_{SL(n)}(u) = 6 \ \ and \ \ d_{SL(n)}(v) = 6\}$

Clearly, we have $E(SL(n)) = E^1(SL(n)) \cup E^2(SL(n)) \cup E^3(SL(n))$.

Theorem 3.2. For $n \geq 2$, the Lanzhou index of hexagons with size n is

$$\mathrm{Lz}(SL(n)) = 5670n^4 + 324n^3 - 2646n^2 + 540n.$$

Proof. For $n \geq 2$, we have $|NE_{3,3}| = 6n$, $|NE_{3,6}| = 6n$ and $|NE_{6,6}| = 18n^2 - 12n$. Let $G = SL(n)$. By the definition of Lanzhou index, we have that

$$\begin{aligned}
\mathrm{Lz}(SL(n)) &= \left(\left(|V(G)| - 1\right) M_1(G) - F(G)\right) \\
&= \sum_{v \in NE_{3,3}} \left(\left(3n(5n+1) - 1\right)(d_G(u) + d_G(v)) - (d_G^2(u) + d_G^2(v))\right) \\
&\quad + \sum_{v \in NE_{3,6}} \left(\left(3n(5n+1) - 1\right)(d_G(u) + d_G(v)) - (d_G^2(u) + d_G^2(v))\right) \\
&\quad + \sum_{v \in NE_{6,6}} \left(\left(3n(5n+1) - 1\right)(d_G(u) + d_G(v)) - (d_G^2(u) + d_G^2(v))\right) \\
&= 6n(6(3n(5n+1)-1) - 18) + (18n^2 + 6n)(9(3n(5n+1)-1) - 45) \\
&\quad + (18n^2 - 12n)(12(3n(5n+1)-1) - 72) \\
&= 5670n^4 + 324n^3 - 2646n^2 + 540n.
\end{aligned}$$

4. RESULTS FOR SIERPIŃSKI GRAPHS

Graphs are widely used in topology, psychology, and probability Hinz and Schief (1990), Kaimanovich (2001), Klix and Goede (1967).The sierpiński graphs $S(n,k)$ was introduced by Pisanski et al. in Pisanski and Tucker (2001).

Let $\{1,2,\dots,k\}^n$ be the vertex set of $S(n,k)$, $n \geq 2$, $k \geq 2$. The edge $u_i v_j$ is between two vertices $u_i = (u_1,\dots,u_n)$ and $v_j = (v_1,\dots,v_n)$. If there is an integer $h \in [n]$ such that

(1) $u_i = v_j$, $j = 1,\dots,h-1$;

(2) $u_h = v_h$;

(3) $u_j = v_h, v_j = u_h, for\ j = h+1,\dots,n.$

The Sierpiński graph $S(2,3)$ is shown in Figure 7.

Theorem 4.1. The Lanzhou index of Sierpiński graph $S(n,3)$ (also named the Tower of Hanoi with n disks) $(n \geq 1)$ is $\mathrm{Lz}(S(n,3)) = 9 \cdot 3^{2n} - 51 \cdot 3^n + 72$.

Proof. Let $G = S(n,3)$. Obviously, the degree of vertex v is either 2 or 3. It is clear that $|NV_2| = 3$ and $|NV_3| = 3n - 3$. Then

$$\begin{aligned}\mathrm{Lz}(G) &= \sum_{v\in V(G)} d_{\overline{G}}(v) d_G^2(v) = \sum_{v\in NV_2} d_{\overline{G}}(v) d_G^2(v) + \sum_{v\in NV_3} d_{\overline{G}}(v) d_G^2(v) \\ &= \sum_{v\in NV_2} 2^2(3^n-3) + \sum_{v\in NV_3} 3^2(3^n-4) = 2^2(3^n-3)\cdot 3 + 3^2(3^n-4)(3^n-3) \\ &= 9\cdot 3^{2n} - 51\cdot 3^n + 72.\end{aligned}$$

The Sierpiński gasket graphs are extended versions of the Sierpiński graph. In 2008, Alberto and Anant introduced Sierpiński gasket graph Teguia and Godbole (2006). The sierpiński gasket $S[n,k]$ is obtained by shrinking all bridge edges of $S(n,k)$. Sierpiński gasket $S[3,3]$ is shown in Figure 8.

Figure 7

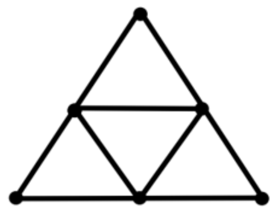

Figure 7 $S(2,3)$

Figure 8

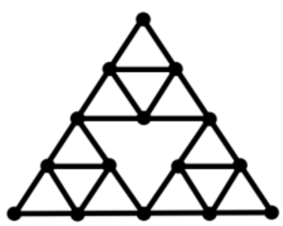

Figure 8 $S[3,3]$

Theorem 4.2. The Lanzhou index of Sierpiński gasket graph $S[n,3] (n \geq 1)$ is

$$\mathrm{Lz}(S[n,3]) = 4\cdot 3^{2n} - 34\cdot 3^n + 66.$$

Proof. In Sierpi´nski gasket graph $S[n,3]$, $|V(S[n,3])| = \frac{3}{2}(3^{n-1}+1)$, $|E(S[n,3])| = 3n$, and the set of edges can decompose as two types:

Type (1): $E^1(S[n,3]) = \{uv | d_{S[n,3]}(u) = 2 \text{and} d_{S[n,3]}(v) = 4\}$

Type (2): $E^2(S[n,3]) = \{uv | d_{S[n,3]}(u) = 4 \text{and} d_{S[n,3]}(v) = 4\}$

Clearly, we have $E(S[n,3]) = E^1(S[n,3]) \cup E^2(S[n,3])$.

Let $G = S[n,3]$, by the definition of Lanzhou index, for $n \geq 2$, we have $|NE_{2,4}| = 6$, $|NE_{4,4}| = .\ \ 3n - 6$. Then we have

$$\mathrm{Lz}(G)=\left(\left(|V(G)|-1\right)M_1(G)-F(G)\right)=\sum_{v\in NE_{2,4}}\left(\left(\left(\frac{3}{2}(3^{n-1}+1)\right)-1\right)(d_G(u)+d_G(v))-(d_G^2(u)+d_G^2(v))\right)$$

$$+\sum_{v\in NE_{4,4}}\left(\left(\left(\frac{3}{2}(3^{n-1}+1)\right)-1\right)(d_G(u)+d_G(v))-(d_G^2(u)+d_G^2(v))\right)$$

$$=6(((\frac{3}{2}(3^{n-1}+1))-1)(2+4)-(2^2+4^2))+(3^n-6)(((\frac{3}{2}(3^{n-1}+1))-1)(4+4)-(4^2+4^2))$$

$$=4\cdot 3^{2n}-34\cdot 3^n+66.$$

Theorem 4.3. The Lanzhou index of Sierpiński gasket graph $S[n,k]$ $(n\geq 1)$ is

$$\mathrm{Lz}(S[n,k])=(k^n-k)(k^{n+2}-3k^2+k).$$

Proof, there are k^n vertices in $S[n,k]$. The number of $k-1$ degree vertex is k, and the number of k degree vertex is k^n-k. Next, we consider the vertex in the complement graph of $S[n,k]$. In $\overline{S[n,k]}$, the number of k^n-1-k degree vertex is k, and there are k^n-k vertices with degree k^n-1-k. Note that $|NV_k|=k^n-k$ and $|NV_{k-1}|=k$. Let $G=S[n,k]$. Then we have

$$\mathrm{Lz}(G)=\sum_{v\in V(G)}d_{\overline{G}}(v)d_G^2(v)=\sum_{v\in NV_{k-1}}d_{\overline{G}}(v)d_G^2(v)+\sum_{v\in NV_k}d_{\overline{G}}(v)d_G^2(v)$$

$$=\sum_{v\in NV_{k-1}}(k-1)^2(k^n-k)+\sum_{v\in NV_k}k^2(k^n-k-1)=k(k-1)^2(k^n-k)+(k^n-k)k^2(k^n-k-1)$$

$$=(k^n-k)(k^{n+2}-3k^2+k).$$

The Sierpiński Gasket Rhombus of level n is defined by $SR(n)$, which obtained by identifying the edges in two copies of $S[n,3]$ along one of their sides and $|V(SR(n))|=3^n-2^{n-1}+2$. Sierpiński Gasket Rhombus $SR(3)$ show in Figure 9.

Figure 9

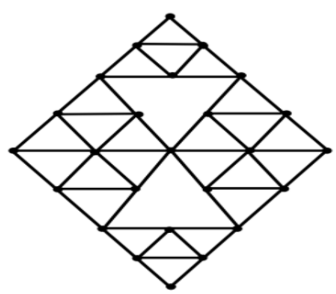

Figure 9 Sierpiński Gasket Rhombus $\boldsymbol{SR(3)}$

Theorem 4.4. If $SR(n)$ is a Sierpiński Gasket Rhombus graph. Then

$$\mathrm{Lz}(SR(n))=3^n(4\cdot 2^{n-1}-110)-2^{n-1}(4\cdot 2^{n-1}+162)+160.$$

Proof. Note that $|NE_{2,4}|=4$, $|NE_{3,4}|=4$, $|NE_{3,6}|=2$, $|NE_{4,4}|=2\cdot 3^n-3\cdot 2^n-4$, $|NE_{6,4}|=4(2^{n-1}-2)$, and $|NE_{6,6}|=2^{2n-1}-2$. Let $SR(n)=G$ and $|V(SR(n))|=N$, clearly, $N=3^n-2^{n-1}+2$. Therefore, we have

$$\begin{aligned}
\mathrm{Lz}(G) &= \left(\left(|V(G)|-1\right)M_1(G)-F(G)\right) = \sum_{u,v\in NE_{2,4}}(N-1)(d_G(u)+d_G(v)) - \sum_{u,v\in NE_{2,4}}(d_G^2(u)+d_G^2(v)) \\
&+ \sum_{u,v\in NE_{3,4}}(N-1)(d_G(u)+d_G(v)) - \sum_{u,v\in NE_{3,4}}(d_G^2(u)+d_G^2(v)) \\
&+ \sum_{u,v\in NE_{6,4}}(N-1)(d_G(u)+d_G(v)) - \sum_{u,v\in NE_{6,4}}(d_G^2(u)+d_G^2(v)) \\
&+ \sum_{u,v\in NE_{6,6}}(N-1)(d_G(u)+d_G(v)) - \sum_{u,v\in NE_{6,6}}(d_G^2(u)+d_G^2(v)) \\
&+ \sum_{u,v\in NE_{3,6}}(N-1)(d_G(u)+d_G(v)) - \sum_{u,v\in NE_{3,6}}(d_G^2(u)+d_G^2(v)) \\
&+ \sum_{u,v\in NE_{4,4}}(N-1)(d_G(u)+d_G(v)) - \sum_{u,v\in NE_{4,4}}(d_G^2(u)+d_G^2(v)) \\
&= 3^n(4\cdot 2^{n-1}-110)-2^{n-1}(4\cdot 2^{n-1}+162)+160.
\end{aligned}$$

5. LANZHOU INDEX OF CACTUS CHAINS NETWORKS

The Cactus chain is a simple connected graph. Husimi and Riddell first studied the Cactus graph in Husimi (1950). These graphs are widely used in many fields such as the theory of electrical and communication networks Zmazek and Zerovnik (2005). and in chemistry Zmazek and Zerovnik (2003).

Theorem 5.1. The Lanzhou index of different types of Cactus graphs.

Use $T_n (n \geq 2)$ represent the chain triangular graph (See Figure 9). Then $\mathrm{Lz}(T_n) = 40n^2 - 88n + 48$.

Use $Q_n (n \geq 2)$ represent the para-chain square cactus graph (See Figure 10). Then $\mathrm{Lz}(Q_n) = 72n^2 - 104n + 48$.

Use $O_n (n \geq 2)$ represent the para-chain square cactus graph (See Figure 11). Then $\mathrm{Lz}(O_n) = 72n^2 - 104n + 48$.

Use $O_n^h (n \geq 2)$ represent the Ortho-chain graph (See Figure 12).Then $\mathrm{Lz}\left(O_n^h\right) = 160n^2 - 136n + 48$

Use $L_n (n \geq 2)$ represent the para-chain hexagonal cactus graph (See Figure 13). Then $\mathrm{Lz}(L_n) = 160n^2 - 136n + 48$.

Use $M_n (n \geq 2)$ represent the Meta-chain hexagonal cactus graph (See Figure 14). Then $\mathrm{Lz}(M_n) = 160n^2 - 136n + 48$.

Figure 10

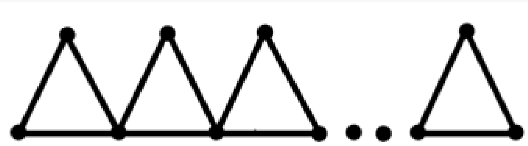

Figure 10 T_n

Figure 11

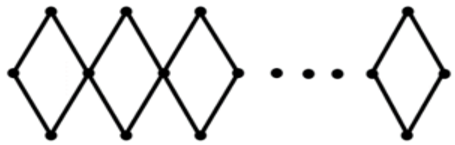

Figure 11 Q_n

Figure 12

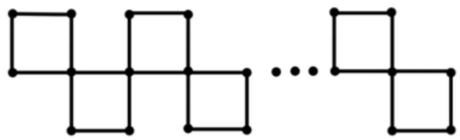

Figure 12 O_n

Figure 13

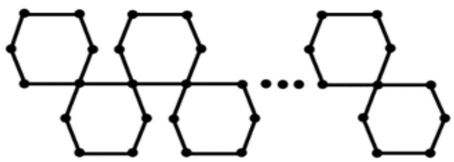

Figure 13 O_n^h

Figure 14

Figure 14 L_n

Figure 15

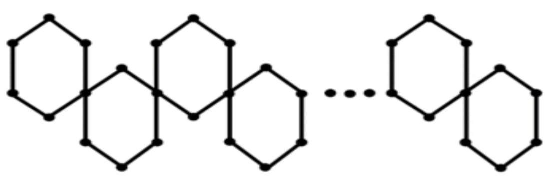

Figure 15 M_n

Proof.

1) Note that $|V(T_n)| = 2n + 1$, $|NE_{2,2}| = 2$, $|NE_{2,4}| = 2n$, and $|NE_{4,4}| = n - 2$. Therefore, we have

$$\begin{aligned}\mathrm{Lz}(T_n) &= \left(\left(|V(T_n)|-1\right)M_1(T_n)-F(T_n)\right) = \sum_{u,v\in NE_{2,2}}(2n+1-1)(d_{T_n}(u)+d_{T_n}(v)) - \sum_{u,v\in NE_{2,2}}(d_{T_n}^2(u)+d_{T_n}^2(v)) \\ &+ \sum_{u,v\in NE_{2,4}}(2n+1-1)(d_{T_n}(u)+d_{T_n}(v)) - \sum_{u,v\in NE_{2,4}}(d_{T_n}^2(u)+d_{T_n}^2(v)) \\ &+ \sum_{u,v\in NE_{4,4}}(2n+1-1)(d_{T_n}(u)+d_{T_n}(v)) - \sum_{u,v\in NE_{4,4}}(d_{T_n}^2(u)+d_{T_n}^2(v)) \\ &= 2((2n+1-1)(2+2)-(2^2+2^2))(n-2)+2n((2n+1-1)(2+4)-(2^2+4^2)) \\ &+((2n+1-1)(4+4)-(4^2+4^2)) \\ &= 40n^2-88n+48.\end{aligned}$$

2) Note that there are four edges with end-vertex of degree 2. Also there are $4n-4$ edges with end-vertex of degree 2 and 4. So we have $|V(Q_n)|=3n+1$, $|NE_{2,2}|=4$ and $|NE_{2,4}|=4n-4$ .Therefore, we have

$$\begin{aligned}\mathrm{Lz}(Q_n) &= \left(\left(|V(Q_n)|-1\right)M_1(Q_n)-F(Q_n)\right) = \sum_{u,v\in NE_{2,2}}(3n+1-1)(d_{Q_n}(u)+d_{Q_n}(v)) - \sum_{u,v\in NE_{2,2}}(d_{Q_n}^2(u)+d_{Q_n}^2(v)) \\ &+ \sum_{u,v\in NE_{2,4}}(3n+1-1)(d_{Q_n}(u)+d_{Q_n}(v)) - \sum_{u,v\in NE_{2,4}}(d_{Q_n}^2(u)+d_{Q_n}^2(v)) \\ &= 4((3n+1-1)(2+2)-(2^2+2^2))+(4n-4)((3n+1-1)(2+4)-(2^2+4^2)) \\ &= 72n^2-104n+48.\end{aligned}$$

3) Note that there are $n+2$ edges with end-vertexs of degree 2, there are $2n$ edges with end-vertexs of degree 2 and 4, and there are $n-2$ edges with end-vertexs of degree 4. Then we have $|V(O_n)| = 3n+1$, $|NE_{2,2}| = n+2$, $|NE_{2,4}| = 2n$ and $|NE_{4,4}| = n-2$. Therefore, we have

$$\begin{aligned}\mathrm{Lz}(O_n) &= \left(\left(|V(O_n)|-1\right)M_1(O_n)-F(O_n)\right) = \sum_{u,v\in NE_{2,2}}(3n+1-1)(d_{O_n}(u)+d_{O_n}(v)) - \sum_{u,v\in NE_{2,2}}(d_{O_n}^2(u)+d_{O_n}^2(v)) \\ &+ \sum_{u,v\in NE_{2,4}}(3n+1-1)(d_{O_n}(u)+d_{O_n}(v)) - \sum_{u,v\in NE_{2,4}}(d_{O_n}^2(u)+d_{O_n}^2(v)) \\ &+ \sum_{u,v\in NE_{4,4}}(3n+1-1)(d_{O_n}(u)+d_{O_n}(v)) - \sum_{u,v\in NE_{2,4}}(d_{O_n}^2(u)+d_{O_n}^2(v)) \\ &= (n+2)(3n(2+2)-(2^2+2^2))+2n(3n(2+4)-(2^2+4^2)) \\ &+(n+2)(3n(4+4)-(4^2+4^2)) \\ &= 72n^2-104n+48.\end{aligned}$$

4) There are $3n+2$ edges with end-vertex of degree 2, there are $2n$ edges with end-vertex of degree 2 and 4, and there are $n-2$ edges with end-vertex of degree 4. Then we have $|V(O_n^h)|=5n+1$, $|NE_{2,2}| = 3n+2$, $|NE_{2,4}| = 2n$ and $|NE_{4,4}| = n-2$. Therefore, we have

$$\begin{aligned}\mathrm{Lz}\left(O_n^h\right) &= \left(\left(\left|V(O_n^h)\right|-1\right)M_1(O_n^h)-F(O_n^h)\right) = \sum_{u,v\in NE_{2,2}}(5n+1-1)(d_{O_n^h}(u)+d_{O_n^h}(v)) - \sum_{u,v\in NE_{2,2}}(d_{O_n^h}^2(u)+d_{O_n^h}^2(v))\\
&+\sum_{u,v\in NE_{2,4}}(5n+1-1)(d_{O_n^h}(u)+d_{O_n^h}(v)) - \sum_{u,v\in NE_{2,4}}(d_{O_n^h}^2(u)+d_{O_n^h}^2(v))\\
&+\sum_{u,v\in NE_{4,4}}(5n+1-1)(d_{O_n^h}(u)+d_{O_n^h}(v)) - \sum_{u,v\in NE_{4,4}}(d_{O_n^h}^2(u)+d_{O_n^h}^2(v))\\
&=(3n+2)((5n+1-1)(2+2)-(2^2+2^2))+2n((5n+1-1)(2+4)-(2^2+4^2))\\
&+(n-2)((5n+1-1)(4+4)-(4^2+4^2))\\
&=160n^2-136n+48.\end{aligned}$$

5) Since $|NE_{2,2}| = 2n+4$ and $|NE_{2,4}| = 4n-4$, $|V(L_n)| = 5n+1$. Then we have

$$\begin{aligned}\mathrm{Lz}\left(L_n\right) &= \left(\left|V(L_n)\right|-1\right)M_1(L_n)-F(L_n) = \sum_{u,v\in NE_{2,2}}(5n+1-1)(d_{L_n}(u)+d_{L_n}(v)) - \sum_{u,v\in NE_{2,2}}(d_{L_n}^2(u)+d_{L_n}^2(v))\\
&+\sum_{u,v\in NE_{2,4}}(5n+1-1)(d_{L_n}(u)+d_{L_n}(v)) - \sum_{u,v\in NE_{2,4}}(d_{L_n}^2(u)+d_{L_n}^2(v))\\
&=(2n+4)((5n+1-1)(2+2)-(2^2+2^2))+(4n+4)((5n+1-1)(2+4)-(2^2+4^2))\\
&=160n^2-136n+48.\end{aligned}$$

6) Since $|V(M_n)| = 5n+1$, $|NE_{2,2}| = 2n+4$ and $|NE_{2,4}| = 4n-4$. Then, we have

$$\begin{aligned}\mathrm{Lz}\left(M_n\right) &= \left(\left|V(M_n)\right|-1\right)M_1(M_n)-F(M_n)\\
&= \sum_{u,v\in NE_{2,2}}(5n+1-1)(d_{M_n}(u)+d_{M_n}(v)) - \sum_{u,v\in NE_{2,2}}(d_{M_n}^2(u)+d_{M_n}^2(v))\\
&+\sum_{u,v\in NE_{2,4}}(5n+1-1)(d_{M_n}(u)+d_{M_n}(v)) - \sum_{u,v\in NE_{2,4}}(d_{M_n}^2(u)+d_{M_n}^2(v))\\
&=(2n+4)((5n+1-1)(2+2)-(2^2+2^2))+(4n+4)((5n+1-1)(2+4)-(2^2+4^2))\\
&=160n^2-136n+48.\end{aligned}$$

6. RESULTS FOR SOME OPERATIONS

For any integer k, the k-subdivision of G is denoted by $G^{\frac{1}{k}}$, which is constructed by replacing each edge of G with a path of length k. Then, we have following result about the Lanzhou index of k-subdivision of graph G.

Theorem 6.1. Let G be a graph with order n size m, $m \geq 1$. For every $k \geq 2$. We have

$$\mathrm{Lz}\left(G^{\frac{1}{k}}\right) = \mathrm{Lz}(G)+4\mathrm{m}(\mathrm{k}\text{-}1)(m(\mathrm{k}\text{-}1)+\mathrm{n}\text{-}3).$$

Proof. There are $d_G(u)$ edges with end-vertex of degree 2 and $d_G(u)$. Also, there are $m(k-2)$ edges with end-vertex of degree 2 and 2. Therefore, we have

$$\mathrm{Lz}\left(G^{\frac{1}{k}}\right)=\left(\left|V(G^{\frac{1}{k}})\right|-1\right)M_1(G^{\frac{1}{k}})-F(G^{\frac{1}{k}})=m(k-2)\left(\left(\left|V(G^{\frac{1}{k}})\right|-1\right)(2+2)-(2^2+2^2)\right)$$
$$+\sum_{u\in V(G)} d_G(u)\left(\left(\left|V(G^{\frac{1}{k}})\right|-1\right)(d_G(u)+2)-(d_G(u)^2+2^2)\right)$$
$$=4m(k-2)(m(k-1)+n-3)+\sum_{u\in V(G)} d_G(u)((m(k-1)+n-1)(d_G(u)+2)-(d_G^2(u)+2^2))$$
$$=4m(k-2)(m(k-1)+n-3)$$
$$+\sum_{u\in V(G)}(-d_G^3(u)+(m(k-1)+n-1)d_G^2(u)+(2(m(k-1)+n)-6)d_G(u))$$
$$=4m(k-2)(m(k-1)+n-3)-\sum_{u\in V(G)} d_G^3(u)+(m(k-1)+n-1)\sum_{u\in V(G)} d_G^2(u)$$
$$+(2(m(k-1)+n)-6)\sum_{u\in V(G)} d_G(u)$$
$$=4m(k-2)(m(k-1)+n-3)-F(G)+(m(k-1)+n-1)M_1(G)$$
$$+2m(2(m(k-1)+n)-6)$$
$$=\mathrm{Lz}(G)+4m(k-1)(m(k-1)+n-3).$$

By the proof of Theorem 13, we get Observation as below.

Observation 2. Let $k\in N$. Then we have

$$\mathrm{Lz}(G^{\frac{1}{k}})=4m(k-1)(m(k-1)+n-3)-F(G)+(m(k-1)+n-1)M_1(G).$$

We next describe some common binary operations defined on graphs. In the following definitions, let G_1 and G_2 are two graphs with disjoint vertex sets. The join $G=G_1\vee G_2$ of G_1 and G_2 has vertex set $V(G)=V(G_1)+V(G_2)$ and edge set

$$E(G)=\cup_{i=1}^{2}E(G_1)\cup\{uv|u\in V(G_1),v\in V(G_2)\}.$$

By the definition of join operation and the definition of Lanzhou index, we have following observation.

Observation 3. Let G and H are two graphs with order n and m, respectively. Then we have

$$\mathrm{Lz}(G\vee H)=\sum_{u\in V(G),v\in V(H)}(|V(G\vee H)|-1)(d_G(u)+m+d_H(v)+n)$$
$$-((d_G(u)+m)^2+(d_H(v)+n)^2)+\sum_{uv\in E(G)}(|V(G\vee H)|-1)(d_G(u)+m+d_G(v)+m)$$
$$-((d_G(u)+m)^2+(d_G(v)+m)^2)+\sum_{uv\in E(H)}(|V(G\vee H)|-1)(d_H(u)+n+d_H(v)+n)$$
$$-((d_G(u)+n)^2+(d_G(v)+n)^2).$$

Theorem 6.2. Let G be a graph with order n and size n_1, and let H be a graph with order m and size m_1. Then we have

$$\mathrm{Lz}(G\vee H)=C-F(G)-F(H)+(m-2n-1)M_1(H)+(n-2m-1)M_1(G).$$

where $C = (m+n-1)(m^2n + mn^2 + 4mn_1 + 4m_1n) - m^3n - mn^3 - 6mn_1 - 6n^2m_1$.

Proof. From Observation 16, we have that

$$\begin{aligned}
\mathrm{Lz}(G \vee H) = & \sum_{u\in V(G), v\in V(H)} (m+n-1)(d_G(u)+m+d_H(v)+n) - \sum_{u\in V(G), v\in V(H)} ((d_G(u)+m)^2 + (d_H(v)+n)^2) \\
& + \sum_{uv\in E(G)} (m+n-1)(d_G(u)+m+d_G(v)+m) - \sum_{u,v\in E(G)} ((d_G(u)+m)^2 + (d_G(v)+m)^2) \\
& + \sum_{uv\in E(H)} (m+n-1)(d_H(u)+n+d_H(v)+n) - \sum_{u,v\in E(G)} ((d_H(u)+n)^2 + (d_H(v)+n)^2) \\
& = (m+n-1) \sum_{uv\in E(H)} (d_H(u)+d_H(v)) + (m+n-1)(m^2n + mn^2 + 2mn_1 + 2m_1n) \\
& - \sum_{u\in V(G), v\in V(H)} (d_G(u)+m)^2 + (d_H(v)+n)^2 + (m+n-1) \sum_{u\in V(G), v\in V(H)} (d_G(u)+d_G(v)) \\
& +(m+n-1) \sum_{uv\in E(G)} (d_G(u)+d_G(v)) - \sum_{uv\in E(G)} ((d_G(u)+m)^2 + (d_G(v)+m)^2) \\
& - \sum_{uv\in E(G)} ((d_H(u)+n)^2 + (d_H(v)+n)^2)
\end{aligned}$$

It is obviously, we have that

$$\begin{aligned}
& \sum_{u\in V(G), v\in V(H)} ((d_G(u)+m)^2 + (d_H(u)+n)^2) \\
& = \sum_{u\in V(G), v\in V(H)} (d_G(u)+m)^2 + \sum_{u\in V(G), v\in V(H)} (d_H(v)+n)^2 \\
& = \sum_{u\in V(G), v\in V(H)} (d_G^2(u) + m^2 + 2md_G(u)) + \sum_{u\in V(G), v\in V(H)} (d_H^2(u) + m^2 + 2md_H(u)) \\
& = mn(m^2+n^2) + \sum_{u\in V(G), v\in (H)} d_G^2(u) + 2m \sum_{u\in V(G), v\in (H)} d_G(u) \\
& + \sum_{u\in V(G), v\in (H)} d_H^2(u) + 2n \sum_{u\in V(G), v\in V(H)} d_H(u) \\
& = mn(m^2+n^2) + mM_1(H) + 4m^2n_1 + 4n^2m_1.
\end{aligned}$$

So, we have following equation.

$$\sum_{u\in V(G), v\in V(H)} ((d_G(u)+m)^2 + (d_H(v)+n)^2) = mn(m^2+n^2) + mM_1(G) + nM_1(H) + 4m^2n_1 + 4n^2m_2$$

Equation 1

It is obviously that

$$\begin{aligned}
\sum_{u\in V(G), v\in V(H)} ((d_G(u)+d_H(u)) &= \sum_{u\in V(G), v\in V(H)} d_G(u) + \sum_{u\in V(G), v\in V(H)} d_H(v) \\
&= \sum_{u\in V(G)} \sum_{v\in V(H)} d_G(u) + \sum_{u\in V(G)} \sum_{v\in V(H)} d_H(u) \\
&= 2(n_1m + m_1n).
\end{aligned}$$

So, we get equation

$$\sum_{u\in V(G),v\in V(H)}((d_G(u)+d_H(u))=2(n_1m+m_1n).$$ Equation 2

It is obviously that

$$\sum_{uv\in E(G)}((d_G(u)+m)^2+(d_G(v)+m)^2)=\sum_{uv\in E(G)}(d_G^2(u)+d_G^2(v)+2m\sum_{uv\in E(G)}(d_G(u)+d_G(v))+2m^2n_1$$
$$=F(G)+2mM_1(G)+2m^2n_1.$$

So, we get equation
$\sum_{uv\in E(G)}((d_G(u)+m)^2+(d_G(v)+m)^2=F(G)+2mM_1(G)+2m^2n_1.$
Equation 3

Similarly, we get equation
$\sum_{uv\in E(G)}((d_H(u)+n)^2+(d_H(v)+n)^2=F(H)+2nM_1(H)+2n^2m_1.$
Equation 4

By equation Equation 1, Equation 2, Equation 3, and Equation 4, we have

$\mathrm{Lz}(G\vee H)=C-F(G)-F(H)+(m-2n-1)M_1(H)+(n-2m-1)M_1(G).$
Where $C=(m+n-1)(m^2n+mn^2+4mn_1+4m_1n)-m^3n-mn^3-6mn_1-6n^2m_1.$

Let G be a graph with order n. The Corona of G and H is defined as the graph obtained by taking one copy of graph G and taking n copy of graph H, then, join ith vertex of G to the vertexs in the i th copy of H.

Observation 4. Let G and H be two graphs, $|V(G)|=n$ and $|V(H)|=m$. Then we have

$$Lz(G\circ H)=\sum_{u\in V(G),v\in V(H)}((n+mn-1)(d_G(u)+m+d_H(v)+1))-((d_G(u)+m)^2+(d_H(v)+1)^2)$$
$$+\sum_{uv\in E(G)}(n+nm-1)(d_G(u)+m+d_G(v)+m)-((d_G(u)+m)^2+(d_G(v)+m)^2)$$
$$+\sum_{uv\in E(H)}(n+nm-1)(d_H(u)+1+d_H(v)+1)-((d_H(u)+1)^2+(d_H(v)+1)^2).$$

By Observation 18, we have result as below.

Theorem 6.3. Let G be a graph with order n and size n_1, and let H be a graph with order m and size m_1. Then we have Let

$\mathrm{Lz}(G \circ H) = C + (n + mn - 3m - 1)M_1(G) + (mn - 3)M_1(H) - F(G) - F(H)$.

where

$C = (n + mn - 1)(4n_1m + 2m_1n + (m+1)nm + 2m_1) - 6m^2n_1 - 4nm_1 - (m^2+1)mn - 2m_1$.

Proof. By Observation 18, it follows that

$$\begin{aligned} Lz(G \circ H) = & \sum_{u \in V(G), v \in V(H)} ((n + mn - 1)(d_G(u) + m + d_H(v) + 1)) - ((d_G(u) + m)^2 + (d_H(v) + 1)^2) \\ & + \sum_{uv \in E(G)} (n + nm - 1)(d_G(u) + m + d_G(v) + m) - ((d_G(u) + m)^2 + (d_G(v) + m)^2) \\ & + \sum_{uv \in E(H)} (n + nm - 1)(d_H(u) + 1 + d_H(v) + 1) - ((d_H(u) + 1)^2 + (d_H(v) + 1)^2). \end{aligned}$$

It obviously that

$$\begin{aligned} & \sum_{u \in V(G), v \in V(H)} ((n + mn - 1)(d_G(u) + m + d_H(v) + 1)) \\ & = (n + mn - 1)\Big(\sum_{u \in V(G)} \sum_{v \in V(H)} (d_G(u) + m + d_H(v) + 1)\Big) \\ & = (n + mn - 1)\Big(\sum_{u \in V(G)} \sum_{v \in V(H)} d_G(u) + \sum_{u \in V(G)} \sum_{v \in V(H)} d_H(v) + \sum_{u \in V(G)} \sum_{v \in V(H)} (m + 1)\Big) \\ & = (n + mn - 1)(2n_1m + 2m_1n + (m + 1)nm). \end{aligned}$$

So, we have following equation

$$\sum_{u \in V(G), v \in V(H)} ((n + mn - 1)(d_G(u) + m + d_H(v) + 1)) = (n + mn - 1)(2n_1m + 2m_1n + (m + 1)nm).$$

Equation 5

Since that

$$\begin{aligned} & \sum_{u \in V(G), v \in V(H)} ((d_G(u) + m)^2 + (d_H(u) + 1)^2) \\ & = \sum_{u \in V(G), v \in V(H)} (d_G(u) + m)^2 + \sum_{u \in V(G), v \in V(H)} (d_H(v) + 1)^2 \\ & = \sum_{u \in V(G), v \in V(H)} (d_G^2(u) + m^2 + 2md_G(u)) + \sum_{u \in V(G), v \in V(H)} (d_H^2(u) + m^2 + 2md_H(u)) \\ & = \sum_{u \in V(G), v \in (H)} (m^2 + 1) + \sum_{u \in V(G), v \in (H)} d_G^2(u) + 2m \sum_{u \in V(G), v \in (H)} d_G(u) + \sum_{u \in V(G), v \in (H)} d_H^2(u) + 2 \sum_{u \in V(G), v \in V(H)} d_H(u) \\ & = mn(m^2 + 1) + mM_1(G) + nM_1(H) + 4m^2n_1 + 4nm_1. \end{aligned}$$

So, it follows that

$$\sum_{u \in V(G), v \in V(H)} ((d_G(u) + m)^2 + (d_H(u) + 1)^2) = mn(m^2 + 1) + mM_1(G) + nM_1(H) + 4m^2n_1 + 4nm_1.$$

Equation 6

Since that

$$\sum_{uv\in E(G)}(n+mn-1)(d_G(u)+m+d_G(v)+m)=(n+mn-1)(M_1(G)+2mn_1).$$

So, it follows that

$$(n+mn-1)\sum_{uv\in E(G)}(d_G(u)+m+d_G(v)+m)=(n+mn-1)(M_1(G)+2mn_1).$$

Equation 7

Since that

$$\begin{aligned}&\sum_{uv\in E(G)}((d_G(u)+m)^2+(d_G(u)+m)^2)\\&=\sum_{uv\in E(G)}\left((d_G^2(u)+2md_G(u)+m^2)+(d_G^2(v)+2md_G(v)+m^2)\right)\\&=\sum_{uv\in E(G)}(d_G^2(u)+d_G^2(v))+2m\sum_{uv\in E(G)}(d_G(u)+d_G(v))+\sum_{uv\in E(G)}2m^2\\&=F(G)+2mM_1(G)+2m^2n_1.\end{aligned}$$

So, it follows that

$$\sum_{uv\in E(G)}((d_G(u)+m)^2+(d_G(u)+m)^2)=F(G)+2mM_1(G)+2m^2n_1.$$

Equation 8

Since that

$$\begin{aligned}\sum_{uv\in E(H)}(n+mn-1)(d_H(u)+1+d_H(v)+1)&=(n+mn-1)\sum_{uv\in E(H)}(d_H(u)+d_H(v))+(n+mn-1)\sum_{uv\in E(H)}2\\&=(n+mn-1)M_1(G)+2(n+mn-1)m_1.\end{aligned}$$

So, it follows that

$$\sum_{uv\in E(H)}(n+mn-1)(d_H(u)+1+d_H(v)+1)=(n+mn-1)M_1(H)+2(n+mn-1)m_1.$$

Equation 9

Since that

$$\begin{aligned}\sum_{uv\in E(H)}((d_H(u)+1)^2+(d_H(u)+1)^2)&=\sum_{uv\in E(G)}\left((d_H^2(u)+2d_H(u)+1)+(d_H^2(v)+2d_H(v)+1)\right)\\&=F(H)+2M_1(H)+2m_1.\end{aligned}$$

So, it follows that

$$\sum_{uv\in E(H)}((d_H(u)+1)^2+(d_H(u)+1)^2)=F(H)+2M_1(H)+2m_1.$$

Equation 10

By Equation 5, Equation 6, Equation 7, Equation 8, Equation 9, and Equation 10, we have that

$$\mathrm{Lz}(G\circ H)=C+(n+mn-3m-1)M_1(G)+(mn-3)M_1(H)-F(G)-F(H).$$

where $C=(n+mn-1)(4n_1m+2m_1n+(m+1)nm+2m_1)-6m^2n_1-4nm_1-(m^2+1)mn-2m_1$.

CONFLICT OF INTERESTS

None.

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