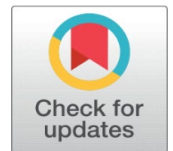# q-SPACE (q-TIME)-DEFORMATION OF THE CONTINUITY EQUATION

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ABSTRACT

A q-space (q-time)-deformation of the continuity equations are introduced using the q-derivative (or Jackson derivative). By quantum calculus, we solve such equations. The free cases are discussed separately.

Keywords: q-Derivative, q- Space Deformation Continuity Equation, q-Time Deformation Continuity Equation, q-Calculus

1. INTRODUCTION

Generally speaking, Continuity equations can comprise both "source" and "sink" items, which permit them to give an account of amounts that are almost preserved, such as the density of sorts of molecules which can be produced or demolished by chemical interactions.

That is to say Continuity equations underlie more specific transfer equations like Navier–Stokes equations, Boltzmann transport equation, and the convection spread equation.

By the diversity theory, a comprehensive continuity equation can also be shaped in a" variant form":

$$\frac{\partial \rho}{\partial t} = \nabla \cdot j = \sigma$$

σ is the generation of Q per unit volume per unit time, it is time; j is the flux of Q, ρ is the amount of the quantity Q per unit volume where ∇ is divergence
When Q is a preserved quantity that cannot be formed or ruined (such as energy), $\sigma = 0$ and the equations become:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

In the field of Mathematics, a q-analog of a theory, sameness or manifestation is an act of generalizing, counting another parameter q that returns the original theory, expression in the limit as q → 1. Mathematicians are, typically, interested in q-analogs that rise naturally, rather than in randomly devising q-analogs of known results. The earliest q-analog dealt with in detail is the basic hyper geometric series, which was presented in the 19th century.

In this way, it is normal to ask what is the q-Space (q-time)-deformation of the continuity equation.

The present paper has been arranged as follows:

Section 1 introduces the topic theory. **Section 2** briefly will give an exposition of some basic concept and preliminaries of the language of q-calculus. **In section 3**, we introduced free q-Space (free q-time)-continuity equation, and we deduced some theorems. Finally, in Section 4, we study q-Space (q-time)-deformation of the continuity equation.

In future works, we will try to extend the continuity equation in the quantum white noise setting, see [8 and 9]

2. PRELIMINARIES

We will give an exposition of some basic concept and preliminaries of the language of q-calculus (see [1, 2, 3, 4, 5, 6, and 7]). Therefore, natural number n has the following q deformation:

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}, \text{ with } [0]_q = 0.$$

Sporadically we will write $[\infty]_q$ for the limit of these numbers: $1/((1-q))$. The q factorials and q binomial coeffcients are defined naturally as

$$[n]_q! := [1]_q \cdot [2]_q \cdots [n]_q \text{ with } [0]_q := 1.$$

For $q \in (0,1)$ and analytic $f:C \rightarrow C$ define operators Z and $D_{(q)}$ as follows:

$$(Z\,f)(z) := (z\,f(z)),$$

$$\left(D_q f\right)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{z(1-q)}, & z \neq 0 \\ \\ f'(0). \end{cases}$$

The probability distribution of a non-commutative random variable $a + a^*$, where a is a bounded operator on some Hilbert space satisfying, the distribution probability of a non-commutative arbitrary changeable $a + a^*$,

$$aa^* + qa^*a = 1, \text{ for some } q \in [-1,1). \tag{1}$$

3. FREE-SPACE (FREE-TIME)-CONTINUITY EQUATION

In this section, the free-Space (Free-Time)-continuity equation cases are discussed.

3.1. FREE-SPACE-CONTINUITY EQUATION

Let D_t given by

$$D_x\big(f(t,x)\big) = \big(Df(t,\cdot)\big)(x) = \left(\frac{f(t,x) - f(t,0)}{x}\right). \tag{2}$$

As a free analogue of the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

we will study the following equation

$$\frac{\partial}{\partial t}\rho = -D_x j. \tag{3}$$

Theorem 1: The free-space-continuity equation (3) gives

$$j(t,x) = j_0(x) - x\frac{\partial}{\partial t}\big(\rho(t,x)\big).$$

Where $\rho(t,\cdot) \in \mathcal{H}^2$ and $\lim_{t\to 0} j(t,\cdot) = j_0 \in \mathcal{H}^2$ are given.

Proof: Using (2) in equation (3), we get

$$\frac{j(t,x) - j(t,0)}{x} = -\frac{\partial}{\partial t}\big(\rho(t,x)\big). \tag{4}$$

Then

$$j(t,x) - j(t,0) = -x\frac{\partial}{\partial t}\rho(t,x).$$

Therefore, we obtain

$$j(t,x) = j_0(x) - x\frac{\partial}{\partial t}\rho(t,x).$$

Which complete the proof.

3.2. FREE-TIME-CONTINUITY EQUATION

Let D_t given by

$$D_t f(t,x) = \big(Df(\cdot,x)\big)(t) = \frac{f(t,x) - f(0,x)}{t}.$$ (5)

As a free analogue of the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

we will study the following equation

$$D_t \rho = -\frac{\partial}{\partial x} j.$$ (6)

Theorem 2. The free-time-continuity equation (6) gives

$$\rho(t,x) = \rho_0(x) - t \frac{\partial}{\partial x} j(t,x),$$

where $j(t,\cdot) \in \mathcal{H}^2$ and $\lim\limits_{t \to 0} \rho(t,\cdot) = \rho_0 \in \mathcal{H}^2$ are given.

Proof: Using (5) in equation (6), we get

$$\frac{\rho(t,x) - \rho(0,x)}{t} = -\frac{\partial}{\partial x} j(t,x).$$

And then

$$\rho(t,x) - \rho(0,x) = -t \frac{\partial}{\partial x} j(t,x).$$

Therefore, we obtain

$$\rho(t,x) = \rho_0(x) - t \frac{\partial}{\partial x} j(t,x)$$

which completes the proof

4. q-SPACE-DEFORMATION OF THE CONTINUITY EQUATION

Let $q \in (0,1)$, Recall that

$$D_{q,x} f(t,x) = \frac{f(t,x) - f(t,qx)}{x(1-q)} \tag{7}$$

As a q-space-deformation of the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

we will study the following equation

$$\frac{\partial}{\partial t} \rho = -D_{q,x} j. \tag{8}$$

Theorem 3: For $q \in (0,1)$, the q-space-deformation of the continuity equation (8) gives

$$j(t,x) = j_0(x) - x(1-q) \sum_{i=0}^{\infty} q^i \frac{\partial}{\partial t} \rho(t, q^i x)$$

Where $\rho(t,\cdot) \in \mathcal{H}^2$ and $\lim_{t \to 0} j(t,\cdot) = j_0 \in \mathcal{H}^2$ are given.

Proof: From equation (7), we get

$$\frac{j(t,x) - \rho(t,qx)}{x(1-q)} = -\frac{\partial}{\partial t} \rho(t,x) \tag{9}$$

And, then

$$\frac{j(t,qx) - \rho(t,q^2x)}{x(1-q)} = -\frac{\partial}{\partial t} \rho(t,qx).$$

$$.$$
$$.$$
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$$\frac{j(t,q^{k-1}x) - \rho(t,q^k x)}{q^{k-1}x(1-q)} = -\frac{\partial}{\partial t} \rho(t, q^{k-1}x)..$$

Then, we obtain

$$j(t,x) - j(t,qx) = -x(1-q)\frac{\partial}{\partial t}\rho(t,x).$$

$$j(t,qx) - j(t,q^2x) = -qx(1-q)\frac{\partial}{\partial t}\rho(t,qx).$$

$$.$$
$$.$$
$$.$$

$$j(t,q^{k-1}x) - j(t,q^kx) = -q^{k-1}x(1-q)\frac{\partial}{\partial t}\rho(t,q^{k-1}x).$$

Therefore, we deduce that

$$j(t,x) - j(t,q^kx) = -x(1-q)\sum_{i=0}^{k-1}q^i\frac{\partial}{\partial t}\rho\big(t,q^ix\big).$$

As k→∞, we get

$$j(t,x) = j_0(x) - x(1-q)\sum_{i=0}^{\infty}q^i\frac{\partial}{\partial t}\rho(t,q^ix)$$

which completes the proof.

5. q-TIME-DEFORMATION OF THE CONTINUITY EQUATION

Let q∈ (0,1). Recall that

$$D_{q,t}f(t,x) = \frac{f(t,x)-f(qt,x)}{t(1-q)} \tag{10}$$

as a q-time-deformation of the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot j = 0$$

we will study the following equation

$$D_{q,t}\rho = -\frac{\partial}{\partial x}j. \tag{11}$$

Theorem 4. For $q \in (0,1)$, the q -time-deformation of the continuity equation (2) gives

$$\rho(t,x) = \rho_0(x) - t(1-q) \sum_{i=0}^{\infty} q^i \frac{\partial}{\partial x} j(q^i t, x)$$

Where $j(t,\cdot) \in \mathfrak{H}^2(\mathfrak{I}_q, \mu_q)$ and $\lim_{t\to 0} \rho(t,\cdot) = \rho_0 \in \mathfrak{H}^2(\mathfrak{I}_q, \mu_q)$ are given.

Proof: From equation (10), we get

$$\frac{\rho(t,x) - \rho(qt,x)}{t(1-q)} = -\frac{\partial}{\partial x} j(t,x) \tag{12}$$

And, then

$$\frac{\rho(qt,x) - \rho(q^2 t,x)}{qt(1-q)} = -\frac{\partial}{\partial x} j(qt,x)$$

$$.$$
$$.$$
$$.$$

$$\frac{\rho(q^{k-1}t,x) - \rho(q^k t,x)}{q^{k-1}t(1-q)} = -\frac{\partial}{\partial x} j(q^{k-1}t,x).$$

Then, we obtain

$$\rho(t,x) - \rho(qt,x) = -t(1-q) \frac{\partial}{\partial x} j(t,x)$$

$$\rho(qt,x) - \rho(q^2 t,x) = -qt(1-q) \frac{\partial}{\partial x} j(qt,x)$$

$$.$$
$$.$$
$$.$$

$$\rho(q^{k-1}t,x) - \rho(q^k t,x) = -q^{k-1}t(1-q) \frac{\partial}{\partial x} j(q^{k-1}t,x).$$

Therefore, we deduce that

$$\rho(t,x) - \rho(q^k t,x) = -t(1-q) \sum_{i=0}^{k-1} q^i \frac{\partial}{\partial x} j(q^i t,x).$$

As $k \to \infty$, we get

$$\rho(t,x) = \rho_0(x) - t(1-q) \sum_{i=0}^{\infty} q^i \frac{\partial}{\partial x} j(q^i t, x)$$

which complete the proof.

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