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A VENDOR-BUYER SUPPLY CHAIN MODEL WITH IMPERFECT PRODUCTION UNDER TIME, PRICE AND PRODUCT RELIABILITY DEPENDENT DEMAND

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ABSTRACT

This article investigates a single-vendor single-buyer supply chain model where the market demand depends on time as well as selling price and product reliability. The vendor's production rate is not constant but depends on the market demand. The vendor's production process is not perfectly reliable; it may produce some percentage of defective items during a production run. The vendor takes up a lot-for-lot policy for delivering the ordered quantity to the buyer who performs 100% screening after receiving each lot. The average total profit of the integrated supply chain is derived and a numerical example is taken to validate the developed model. The optimal results of the proposed model are also discussed for some particular cases. Sensitivity analysis is performed to investigate the influence of key model-parameters on the optimal results.

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1. INTRODUCTION

Over the past several decades, the integration of the individual mind-sets of the vendor and the buyer into the supply chain has been a matter of curiosity to many supply chain researchers. This helps to identify problem areas in the process, allows businesses to take decisive steps and reduce costs to improve on final prices. Improving end-customer gratification and reliability is a by-product of an integrated supply chain as customer's perception improves on-time delivery. The integrated policy makes the supply chain more transparent, making buyers and vendors more flexible and progressive in relation to each other and to the market. Undoubtedly the market demand plays a vital role in determination of such an integrated policy.

There are many products for which the market demand may increase with the passage of time. However, there are many other factors like product price, after sales service, advertisement, product quality, etc. which can also affect the market demand. When a customer wants to buy a product from a shopping mall or supermarket, two things are knocked in his/her mind. What is the price of the product? How reliable the product is? Price and reliability play important roles in the minds of strong customers while buying a product like mobile phone or laptop. Apple is outstanding for its superior quality but it is so expensive that it always remains beyond the reach of the middle class family. It only influences high class customers. On the other hand, Samsung and Lenovo are famous for products that reach relatively low prices and features. Consumer's preference for higher prices to quality will influence himself to buy Apple products which are regarded suitable for their requirements. By preferring features and prices, end-customers' accommodate on product quality and buy Samsung or Lenovo products. The vendors need both advanced manufacturing processes and good quality raw materials if they want to produce high reliability products. To assemble laptops, Lenovo uses low cost raw-materials to reduce overall costs whereas Apple uses very high quality ingredients in MacBooks and hence overall costs increases. To optimize a firm's profit, a balance needs to be struck between time, price and reliability of its product.

Integrated supply chain models have been developed in the literature based on some limited assumptions. One of these assumptions is that the vendor produces products of perfect quality. In fact, there can be a few imperfect items in any production lot due to poor control of processes, non-adherence to plans, inappropriate operating guidelines, and so on. If the vendor has to pay extra cost for each defective item produced then it is profitable to reduce the number of defective items in the production process. The rate of defective items produced by the vendor affects other critical decisions such as the vendor's production lot size and reliability of the product. Further, a vendor has a reputation for making more reliable product which is preferable for a buyer to place an order. To improve the quality of a product, investment can be made to reduce errors in the vendor's production process. In an integrated supply chain system, when non-conformable items are produced, it is most likely that some kind of supervision/inspection activity needs to be performed by the buyer before selling the goods to the end customers.

This article develops an integrated single-vendor single-buyer supply chain model with time, price and reliability dependent market demand. The vendor's production process is imperfect and it rejects all the non-conformable items produced during a production run. The buyer screens all the items before selling to end-customers. The defective items are sold in the secondary market with a discount. The vendor plans for a lot-for-lot production policy to meet the buyer's demand. The primary objectives of this article are to find the response of the following queries:

1) How much time will be taken by the vendor and the buyer to produce a lot and sell to customers?
2) How much time will be delayed by the vendor to produce items ordered by the buyer?
3) What will be the selling price of each good item from the buyer's side?
4) What will be the reliability of a product produced by the vendor?

The rest of the paper is arranged as follows: In the next section, the related literatures are reviewed. Section 3 presents assumptions and notations for developing the proposed model. Section 4 discusses the mathematical model and solution procedure. A numerical example is provided in Section 5. The optimal results are analyzed in Section 6. Section 7 concludes the paper and indicates some future research directions.

2. LITERATURE REVIEW

In reality, the market demand of certain products may not remain constant always; it may change with the passage of time. Hariga and Benkherouf (1994) presented a heuristic inventory model in which the market demand changes exponentially in time over a finite planning horizon. Hariga (1996) developed an inventory lot-sizing model with time-varying demand for deteriorating items. An inventory model with Weibull deterioration, time proportional demand rate and effects of inflation was developed by Chen (1998). Khanra and Chaudhuri (2003) proposed an inventory model with quadratic time dependent demand where the on-hand inventory deteriorates with time. Ghosh and Chaudhuri (2006) developed this model by considering shortages in inventory. Actually, a large volume of research papers on time dependent demand are available in the literature Giri and Maiti (2012), Chowdhury et al. (2014), Samanta et al. (2018)

Now-a-days the customer's demand depends not only on time but also on other factors such as product price, after sales service, advertisement, product quality, etc. Price of a product plays an important role in customer's mind. So, it is more realistic to include price sensitive demand. Burwell et al. (1991) determined the optimal lot size and selling price when a supplier offers all-unit quantity discounts by considering price-dependent demand and allowing for shortages. A finite period system was considered by Datta and Paul (2001) under multi-replenishment scenario, where the demand rate is influenced by both displayed stock level and selling price. An economic production quantity (EPQ) model for deteriorating items was developed by Teng and Chang (2005) where the demand rate depends on the selling price and display stock level with limited display space consideration. You (2005) investigated a supply chain model in which a leading member of the supply chain gets the scope to settle value of the product to impress demand and more revenues. Avinadav et al. (2013) formulated a model for finding the optimal pricing, order quantity and replenishment period for deteriorating items with price- and time-dependent demand. Yang et al. (2013) studied a piecewise production-inventory model for a deteriorating item with time-varying and price-sensitive demand to optimize the vendor's total profit. Herbon and Khmelnitsky (2017) considered a dynamic pricing policy for perishable products, attracting customers to buy less-fresh products due to expiry, potentially increasing revenue and eliminating waste. Numerous works in this direction could be found in the literature You and Hsieh (2007), Chen et al. (2010), Ghosh et al. (2011), Kim et al. (2011), Bhunia and Shaikh (2014), Maiti and Giri (2015), Giri and Roy (2015), Maiti and Giri (2017), Chan (2019), Roy and Giri (2020).

When end-customers buy some goods from buyers, it is the outcome of the endeavors of several members of supply chains. But, the main credit goes to the vendor as the customer prefer that product for his reliability. So, the balance between price and reliability is an important factor in inventory/supply chain management. Therefore, the reliability of a product must be taken into consideration. An EPQ model with a flexible and imperfect production process was proposed by Cheng (1989) under reliability consideration. Sadjadi et al. (2009) considered a production-marketing problem where the reliability of the production process assumed to be imperfect and the inventory and the setup costs per production cycle are not known in advance. An inventory model with imperfect production process was developed by Shah and Shah (2014) for time-declining demand pattern where reliability of the production process was considered as a decision variable. Shah and Vaghela (2018) analysed EPQ model with time and advertisement sensitive demand with the effect of inflation and reliability.

The above works considered reliability of the product and its effect on the optimal results. However, none of these works would consider the market demand as a function of reliability of the product. Khara et al. (2017) considered a model that deals with an imperfect production process, where both perfect and imperfect quality items are produced and demand depends on selling price and reliability of the product. Later, Khara et al. (2019) developed that model by considering demand as a function of selling price, reliability of the product and advertisement cost. Shah and Naik (2020) investigated an inventory model with imperfect production process and reliability-dependent demand.

Chung and Wee (2008) developed an integrated production-inventory deteriorating model considering imperfect production, inspection planning and warranty-period-and stock-level-dependant demand. Jauhari (2016) proposed a vendor-buyer model where the lot transferred from the vendor to the buyer contains some defective items and the buyer conducts an imperfect inspection process to classify the quality of the items. Jauhari et al. (2016) developed an imperfect production-inventory model where the buyer uses periodic review policy to manage his inventory. The demand on the buyer side was assumed to be normally distributed, and the shortage was assumed to be fully backordered and the defective rate of the items was assumed to be fixed.

In this article, we consider the market demand as a function of time, selling price and reliability of the product. The production rate is not constant but depends on the market demand, as considered by Giri and Maiti (2012). The variable production rate was also considered by Jauhari et al. (2016). In the literature, unit production cost is considered as a fixed. But in reality, it should depends on order quantity to be produced by the vendor. More production implies less unit production cost and less production implies expensive production cost. On the other hand, if a vendor prefers to produce an item with more reliable to keep/increase his reputation in market, then (s)he has to use raw material which are also more reliable. Thus the material cost depends on reliability of the product. The demand may change at any time during production process. In that case, to maintain the on-time delivery to the buyer, the vendor's production rate has to be changed. Therefore, we consider the unit production cost as a function of material cost and production rate. Variable unit production cost was also considered in different forms by Khara et al. (2017).

3. MODEL ASSUMPTIONS AND NOTATIONS

The notations used throughout the paper are as follows:

T	:	time interval between successive deliveries (decision variable)
t_m	:	time delayed by the vendor to start production (decision variable)
p	:	unit selling price for the buyer(decision variable)
R	:	reliability of the product (decision variable)
t	:	variable time
n	:	number of cycles
$D(t,p,R)$	:	demand rate at the buyer
$P(T,t,p,R)$	:	production rate at the vendor ($P > D$)

k_1	:	scaling constant for production rate
S_v	:	set up cost per production run for the vendor
S_b	:	ordering cost per order for the buyer
h_v	:	unit stock-holding cost per unit per unit time for the vendor
h_b	:	unit stock-holding cost per unit per unit time for the buyer
Q_n	:	quantity produced by the vendor during the period $[(n-1)T+t_m, nT]$
D_n	:	market demand during the period $[nT,(n+1)T]$
$M(R)$	:	material cost
α	:	price elasticity to demand
β	:	reliability elasticity to demand
γ	:	reliability elasticity to material cost
M_0	:	fixed material cost
M_1	:	material cost increases the reliability of the produced item
k_2	:	variation constant of tool/die costs
F	:	transportation cost per shipment
$C(t,p,R)$	:	unit production cost
z	:	screening rate
d	:	unit screening cost
W	:	unit wholesale price for the vendor
w	:	discount price per defective item for the vendor
I_b	:	buyer's inventory level
I_v	:	vendor's inventory level
Π_b	:	buyer's profit function
Π_v	:	vendor's profit function
Π	:	average total profit to the whole supply chain

The following assumptions are made to develop the proposed integrated vendor-buyer inventory model:

- The supply chain consists of a single-vendor and a single-buyer who stocks and sells a single product.
- The demand for a product depends on time (t), selling price (p) as well as the reliability of the product (R). We assume that the demand rate $D(t,p,R)=(a+bt)p^{-\alpha}(1-R)^{-\beta}$; $a\geq 0, b\geq 0$ and $\alpha>0,\beta>0$ are real constants. This type of demand was considered by Khara et al. (2017).
- The vendor follows the lot-for-lot policy for replenishment made to the buyer.
- The buyer receives the first order from the vendor at time T and (s)he receives order from the vendor in every T time interval.
- Shortages are not allowed in the buyer's inventory.
- As the reliability of the product depends not only on the manufacturing system but also on the quality of the raw material of the product, we assume that the material cost $M(R)$ is an increasing function of the reliability (R) of the product such that $M(R)=M_0+M_1(1-R)^{-\gamma}$, where $M_0>0, M_1>0$ and $\gamma>0$.

- The production rate of the vendor varies with the demand rate. Also, the production rate is greater than the demand rate. We take the production rate $P(t,p,R)$ as $P(t,p,R) = k_1.D(T+t,p,R)$ where $k_1 > 1$.
- As the vendor's production rate is greater than the buyer's demand rate, the vendor may start production with a time delay (t_m) in the n-th production cycle.
- The production cost not only depends on the material cost $M(R)$ but also on tool or die cost, which is proportional to the vendor's production rate. Therefore, the unit production cost $C(t,p,R)$ is assumed as $C(t,p,R) = M(R) + k_2 P(t,p,R)$, where $k_2 > 0$.
- The vendor's production process is not perfectly reliable. During a production run, it may produce some defective (non-conforming) items.
- The buyer starts error-free screening after received products from vendor. We assume that the number of perfect units is at least equal to the demand during the screening time.
- Product quality may be imperfect. In other words, only R% of all produced items meet the demand while $(1-R)$% of items are defective. It is apparent that the maximum reliability of the production process cannot exceed 1. This type of assumption was also considered by Sadjadi et al. (2009).
- The vendor produced Q_n quantity in total during n-th production cycle and delivered to the buyer to meet the customer / market demand D_n in the next cycle.

4. MODEL FORMULATION

The graphical presentation of the vendor-buyer model is shown in Figure 1. We suppose that T is the length of each cycle. For the n-th cycle, the vendor starts his/her production at time $(n-1)T + t_m$ and the buyer receives his/her order of quantity Q_n from the vendor at time nT, $n = 1,2,3,\ldots.$ and meets the market demand D_n for period $[nT,(n+1)T]$. The buyer starts screening at a rate of z units per unit time immediately after receiving the products from the buyer. The buyer's screening is completed at time $(nT + \frac{Q_{m,n}}{z})$. We assume that only R% of received products are acceptable as good products to meet the customer demand. The customer's demand rate at time $t \in [nT,(n+1)T]$ is $D(t,p,R) = (a+bt)p^{-\alpha}(1-R)^{-\beta}$ where $a \geq 0, b \geq 0, \alpha > 0$ and $\beta > 0$ are real constants.

Therefore, the total demand during the period $[nT,(n+1)T]$ is given by

$$D_n = \int_{nT}^{(n+1)T} D(t,p,R)dt = T\{a + b(n+\tfrac{1}{2})T\}p^{-\alpha}(1-R)^{-\beta},\ n = 1,2,3,\ldots \quad (1)$$

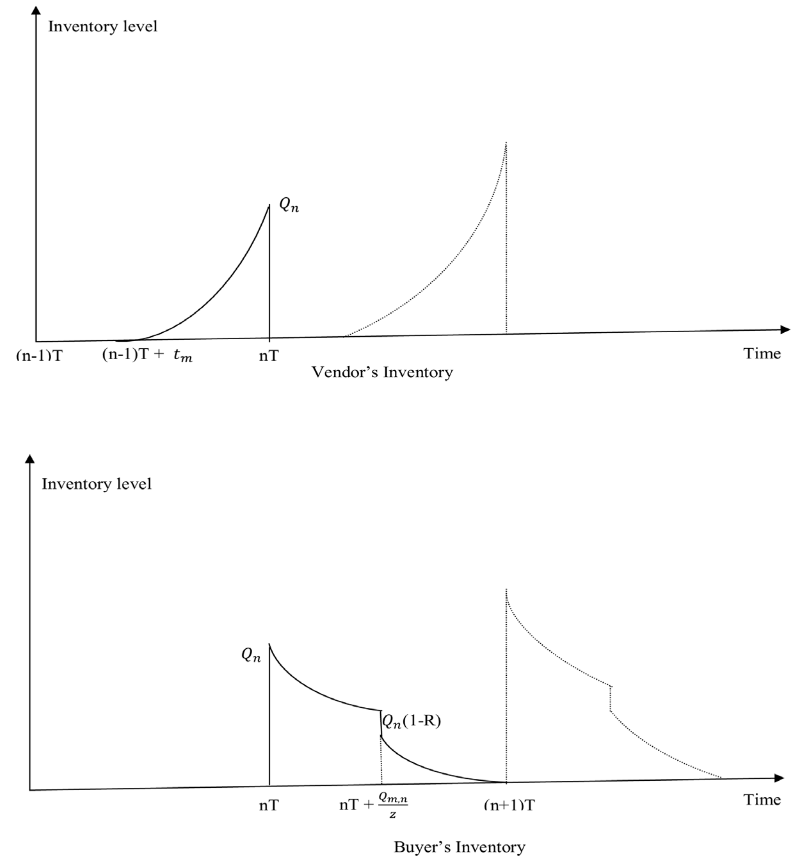


Figure 1 A schematic diagram to represent the vendor's and the buyer's inventory.

The quantity Q_n produced by the vendor in the time interval $[nT, (n+1)T]$ is given by

$$Q_n = \int_{(n-1)T+t_m}^{nT} P(t, p, R)dt$$
$$= k_1 p^{-\alpha}(1-R)^{-\beta}(T-t_m)[a + \frac{b}{2}\{(2n+1)T + t_m\}], \ \ n = 1,2,3, \ldots \quad (2)$$

4.1. DECENTRALISED MODEL

4.1.1. VENDOR'S PERSPECTIVE

Let $I_v(t)$ be the vendor's inventory level at any time $t \in [nT, (n+1)T]$. Then the instantaneous states of the vendor's inventory level can be described by the differential equation:

$$\frac{dI_v(t)}{dt} = P(p, R, t), \ \ (n-1)T + t_m \le t \le nT \ \ \text{ with } I_v((n-1)T + t_m) = 0 \quad (3)$$

Solving (3), we get

$$I_v(t) = k_1[a + \frac{b}{2}\{t + (n+1)T + t_m\}]\{t - ((n-1)T + t_m)\}p^{-\alpha}(1-R)^{-\beta} \quad (4)$$

At time $t = nT$, we have

$$I_v(nT) = k_1 p^{-\alpha}(1-R)^{-\beta}(T - t_m)[a + \frac{b}{2}\{(2n+1)T + t_m\}]$$
$$= Q_n \text{ [using(2)]}$$

The vendor's holding cost per unit time for the period $[(n-1)T, nT]$

$$= \frac{h_v}{T}\int_{(n-1)T+t_m}^{nT} I_v(t)dt$$
$$= \frac{k_1 h_v}{6T}[3a + b\{(3n+1)T + 2t_m\}](T - t_m)^2 p^{-\alpha}(1-R)^{-\beta}$$

The vendor's production cost per unit time in that period

$$= \frac{1}{T}\int_{(n-1)T+t_m}^{nT} C(t,p,R).P(t,p,R)dt$$
$$= \frac{k_1(T-t_m)^2 p^{-\alpha}(1-R)^{-\beta}}{6T}[3\{M_0 + M_1(1-R)^{\gamma}\}\{2a + b\big((2n+1)T + t_m\big)\} + 2k_1k_2p^{-\alpha}$$
$$(1-R)^{-\beta}\{3a^2 + 3ab((2n+1)T + t_m) + b^2((3n^2 + 3n + 1)T^2 + (3n+1)Tt_m + t_m^2)\}]$$

As the vendor's sales revenue = $W\{a + b(n + \frac{1}{2})T\}p^{-\alpha}(1-R)^{-\beta}$, set-up cost = $\frac{S_v}{T}$, discount cost for defective items per unit time = $\frac{wQ_n(1-R)}{T}$, therefore, the vendor's total profit per unit time is given by

$$\Pi_v(t_m, R) = \text{Sales revenue} - \text{holding cost} - \text{production cost} - \text{setup cost} - \text{discount cost}$$

$$= W\{a + b(n + \frac{1}{2})T\}p^{-\alpha}(1-R)^{-\beta} - \frac{wk_1}{T}[a + \frac{b}{2}\{(2n+1)T + t_m\}](T - t_m)p^{-\alpha}(1-R)^{1-\beta}$$

$$-\frac{S_v}{T} - \frac{k_1(T-t_m)^2}{6T}[2k_1k_2p^{-\alpha}(1-R)^{-\beta}\{3a^2 + 3ab\big((2n+1)T + t_m\big) + b^2((3n^2 + 3n + 1)T^2 + (3n+1)Tt_m + t_m^2)\} + h_v\{3a + b\{(3n+1)T + 2t_m\}\} + 3\{M_0 + M_1(1-R)^{\gamma}\}$$
$$\{2a + b((2n+1)T + t_m)\}]p^{-\alpha}(1-R)^{-\beta} \quad (5)$$

4.1.2. BUYER'S PERSPECTIVE

The differential equation governing the buyer's inventory level at any time $t \in [nT, (n+1)T]$is given by

$$\frac{dI_b}{dt} = \begin{cases} -(a+bt)p^{-\alpha}(1-R)^{-\beta} \text{ with } I_b(nT) = Q_n, & nT \le t < (nT + \dfrac{Q_n}{z}) \\ -(a+bt)p^{-\alpha}(1-R)^{-\beta} \text{ with } I_b((n+1)T) = 0, & (nT + \dfrac{Q_n}{z}) \le t \le (n+1)T \end{cases}$$

Solving, we get

$$I_b(t) = \begin{cases} Q_n + (nT - t)\{a + \frac{b}{2}(nT + t)\}p^{-\alpha}(1-R)^{-\beta}, & nT \le t < (nT + \frac{Q_n}{z}) \\ \{(n+1)T - t\}[a + \frac{b}{2}\{(n+1)T + t\}]p^{-\alpha}(1-R)^{-\beta}, & (nT + \frac{Q_n}{z}) \le t \le (n+1)T \end{cases} \quad (6)$$

From (6), the buyer's inventory level at the time point$(nT + \frac{Q_n}{z})$is given by

$$I_b(nT + \frac{Q_n}{z}) = Q_n[R - \{\frac{a}{z} + \frac{bQ_n}{2z^2} + \frac{bnT}{z}\}p^{-\alpha}(1-R)^{-\beta}] \quad (7)$$

Also, we have

$$I_b(nT + \frac{Q_n}{z}) = (T - \frac{Q_n}{z})[a + \frac{b}{2}\{(2n+1)T + \frac{Q_n}{z}\}]p^{-\alpha}(1-R)^{-\beta} \quad (8)$$

From (7) and (8), we have

$$\frac{bQ_n^2}{2z^2} + Rp^{-\alpha}(1-R)^{-\beta}Q_n - T\{a + b(n + \frac{1}{2})T\} = 0 \quad (9)$$

Which is a quadratic equation in Q_n with discriminant

$$R^2p^{2\alpha}(1-R)^{2\beta} + \frac{2bT}{z^2}\{a + b(n + \frac{1}{2})T\} > 0.$$

Hence there always exists a positive (real) production lot size (Q_n) of the vendor in any time interval $[nT, (n+1)T]$, for all $n \in N$.

Now, the buyer's holding cost per unit time

$$= \frac{h_b}{T}\int_{nT}^{(n+1)T} I_b(t)dt$$

$$= \frac{h_b}{12zT} p^{-2\alpha}(1-R)^{-2\beta}[2\{3a + b(3n+2)T\}zT^2 p^{\alpha}(1-R)^{\beta} - 3k_1 T(T-t_m)$$
$$\{4a^2 + 2ab(2(2n+1)T + t_m) + b^2((2n+1)T + t_m)\} + 3k_1^2(T-t_m)^2\{4a^2 + 4ab$$
$$((2n+1)T + t_m) + b^2\{(2n+1)^2T^2 - t_m^2\}\}]$$

Also, sales revenue per unit time = $\{a + b(n+\frac{1}{2})T\}p^{1-\alpha}(1-R)^{-\beta}$, purchase cost per unit time $=W\{a + b(n+\frac{1}{2})T\}p^{-\alpha}(1-R)^{-\beta}$, transportation cost per unit time = $\frac{F}{T}$, screening cost per unit time = $\frac{dk_1}{T}[a + \frac{b}{2}\{(2n+1)T + t_m\}](T - t_m)p^{-\alpha}(1-R)^{-\beta}$ and ordering cost per unit time = $\frac{S_b}{T}$. Therefore, the buyer's total profit per unit time is given by

$$\Pi_b(T,p) = \text{sales revenue} - \text{purchase cost} - \text{holding cost} - \text{transportation cost} - \text{screening cost} - \text{ordering cost}$$

$$= [\{p - W - \frac{dk_1(T-t_m)}{T}\}\{a + \frac{b}{2}(2n+1)T\} - \frac{Th_b}{2}\{a + \frac{b}{3}(3n+2)T\} - \frac{bdk_1t_m}{2T}$$
$$(T-t_m)]p^{-\alpha}(1-R)^{-\beta} - \frac{k_1h_b}{4zT}(T-t_m)[k_1(T-t_m)\{4a^2 + 4ab((2n+1)T + t_m$$
$$+b^2((2n+1)^2T^2 - t_m^2)\} - T\{4a^2 + 2ab(2(2n+1)T + t_m) + b^2((2n+1)T$$
$$+t_m)\}]p^{-2\alpha}(1-R)^{-2\beta} - \frac{F+S_b}{T} \qquad (10)$$

Proposition 1 *When the buyer's selling price p is known, the profit function* $\Pi_b(T,p)$ *is concave with respect to T for all* $T > max\{X_1, X_2, X_3\}$ *where*

$$X_1 = [\frac{3dk_1t_m^2}{(3n+2)h_b}]^{\frac{1}{3}}$$

$$X_2 = [(\frac{dzp^{\alpha}(1-R)^{\beta}}{bh_b} - k_1t_m^2)\frac{t_m}{(2n+1)(k_1-1)}]^{\frac{1}{3}}$$

$$X_3 = [\frac{(4n+3)k_1t_m + 2n+1}{3k_1(2n+1)^2}t_m^3]^{\frac{1}{4}}$$

provided that $0 < t_m < \sqrt{\frac{dzp^{\alpha}(1-R)^{\beta}}{k_1bh_b}}$

Proof. Differentiating (10) twice with respect to T, we get

$$\frac{d^2\Pi_b(T,p)}{dT^2} = -\frac{p^{-2\alpha}(1-R)^{-2\beta}}{6T^3z}[12p^{2\alpha}(1-R)^{2\beta}z(F+S_b) + 2bp^{\alpha}(1 - R)^{\beta}z\{h_b(3n+2)T^3 - 3dk_1t_m^2\} + 12a^2h_bk_1^2t_m^2 + 12ak_1\{bh_b((k_1 - 1)(2n+1)T^3 + k_1t_m^3) - dp^{\alpha}(1-R)^{\beta}t_mz\} + 3b^2h_bk_1\{3k_1(2n + 1)^2T^4 - k_1t_m^4 - (2n+1)T^3 \{1 + 2(2n+1)k_1t_m\}\}]$$

It is clear from the above that $\frac{d^2\Pi_b(T,p)}{dT^2} < 0$ provided that

(i) $h_b(3n+2)T^3 - 3dk_1t_m^2 > 0$ which gives $T > [\frac{3dk_1t_m^2}{(3n+2)h_b}]^{\frac{1}{3}} = X_1$ (say)

(ii) $bh_b((k_1-1)(2n+1)T^3+k_1t_m^3)-dp^\alpha(1-R)^\beta t_m z>0$ which gives

$$(k_1-1)(2n+1)T^3>(\frac{dzp^\alpha(1-R)^\beta}{bh_b}-k_1t_m^2)$$

$$\text{or, } T>[(\frac{dzp^\alpha(1-R)^\beta}{bh_b}-k_1t_m^2)\frac{t_m}{(2n+1)(k_1-1)}]^{\frac{1}{3}}=X_2 \text{ (say)}$$

(iii) $3k_1(2n+1)^2T^4-k_1t_m^4-(2n+1)T^3\{1+2(2n+1)k_1t_m\}>0$

or, $3k_1(2n+1)^2T^4>k_1t_m^4+(2n+1)T^3\{1+2(2n+1)k_1t_m\}$

or, $3k_1(2n+1)^2T^4>k_1t_m^4+(2n+1)t_m^3\{1+2(2n+1)k_1t_m\}$[since $T>t_m$]

or, $3k_1(2n+1)^2T^4>t_m^3\{(2n+1)+(4n+3)k_1t_m\}$

or, $T>[\frac{(4n+3)k_1t_m+2n+1}{3k_1(2n+1)^2}t_m^3]^{\frac{1}{4}}=X_3$(say)

Hence the proposition is proved.

Proposition 2 *For $T>max\{X_4,X_5\}$ where $X_4=\frac{k_1}{k_1-1}t_m$ and $X_5=\frac{1+2(n+1)k_1t_m+\sqrt{(1+2nk_1t_m)^2+4k_1t_m}}{2(2n+1)k_1}$, the profit function $\Pi_b(T,p)$ is concave with respect to p for all p satisfying the condition $W<p<(W+dk_1(1-\frac{t_m}{T}))$.*

Proof. Differentiating (10) twice with respect to p, we get

$$\begin{aligned}\frac{d^2\Pi_b(T,p)}{dp^2}=&-\frac{\alpha}{2}p^{-2(1+\alpha)}(1-R)^{-2\beta}[4p^{1+\alpha}(1-R)^\beta\{a+\frac{b}{2}(2n+1)T\\&+\frac{k_1h_b(T-t_m)(2\alpha+1)}{Tz}\{\{k_1(T-t_m)-T\}\{4a^2+4ab(2n+1)T\}\\&+2abt_m\{2k_1(T-t_m)-T\}+b^2\{(2n+1)T+t_m\}\{k_1(T-t_m)\{(2n\\&+1)T-t_m\}-T\}\}+2(1+\alpha)p^\alpha(1-R)^\beta\{\frac{h_bT}{2}\{a+\frac{b}{3}(3n+2)T\}\\&+\frac{bdk_1t_m(T-t_m)}{2T}+a+\frac{b}{2}(2n+1)T\{W+dk_1(1-\frac{t_m}{T})-p\}\}]\end{aligned}$$

From above, $\frac{d^2\Pi_b(T,p)}{dp^2}<0$ provided that the following conditions hold:

(i) $2k_1(T-t_m)>k_1(T-t_m)>T$ which implies $T>\frac{k_1}{k_1-1}t_m=X_4$.

(ii) $k_1(T-t_m)\{(2n+1)T-t_m\}-T>0$

Considering the above inequation as equation, we see that the two roots of the equation are

$$T=\frac{1+2(n+1)k_1t_m\pm\sqrt{(1+2nk_1t_m)^2+4k_1t_m}}{2(2n+1)k_1}$$

We take T such that$T>\frac{1+2(n+1)k_1t_m+\sqrt{(1+2nk_1t_m)^2+4k_1t_m}}{2(2n+1)k_1}=X_5$.

(iii) $W+dk_1(1-\frac{t_m}{T})-p>0$

As the buyer's selling price p is always greater than the vendor's wholesale price W, we have $W<p<(W+dk_1(1-\frac{t_m}{T}))$. Hence, the proposition is proved.

Proposition 3 *For known R, p and T, the vendor's profit function $\Pi_v(R, t_m)$ is concave with respect to t_m if* $t_m \gtrless \frac{2bk_1k_2(a+bnT)p^{\alpha}(1-R)^{\beta}+b\{M_0+M_1(1-R)^{\gamma}+w(1-R)\}-h_v\{a+b(n-1)T\}}{2b\{h_v-2bk_1k_2p^{-\alpha}(1-R)^{-\beta}\}}$ provided that $h_v \gtrless bk_1k_2p^{-\alpha}(1-R)^{-\beta}$.

Proof. Differentiating (5) twice with respect to t_m, we get

$$\frac{d^2\Pi_v(t_m,R)}{dt_m^2} = -\frac{k_1p^{-2\alpha}(1-R)^{-2\beta}}{T}[p^{\alpha}(1-R)^{\beta}\{h_v\{a+b((n-1)T+2t_m)\} - b\{M_0+M_1(1-R)^{\gamma}+w(1-R)\}\} - 2bk_1k_2\{a+b(nT+t_m)\}]$$

Clearly, $\frac{d^2\Pi_v(t_m,R)}{dt_m^2} < 0$ if $h_v\{a+b((n-1)T+2t_m) > b\{M_0+M_1(1-R)^{\gamma} +w(1-R)\} + 2bk_1k_2\{a+b(nT+t_m)\}p^{-\alpha}(1-R)^{-\beta}$

or, $$2b\{h_v-2bk_1k_2p^{-\alpha}(1-R)^{-\beta}\}t_m > 2bk_1k_2(a+bnT)p^{\alpha}(1-R)^{\beta} + b\{M_0 +M_1(1-R)^{\gamma}+w(1-R)\} - h_v\{a+b(n-1)T\}$$

If $h_v > 2bk_1k_2p^{-\alpha}(1-R)^{-\beta}$, then from above we have,

$$t_m > \frac{2bk_1k_2(a+bnT)p^{\alpha}(1-R)^{\beta}+b\{M_0+M_1(1-R)^{\gamma}+w(1-R)\}-h_v\{a+b(n-1)T\}}{2b\{h_v-2bk_1k_2p^{-\alpha}(1-R)^{-\beta}\}}$$

Again, if $h_v < 2bk_1k_2p^{-\alpha}(1-R)^{-\beta}$, then from above we have,

$$t_m < \frac{2bk_1k_2(a+bnT)p^{\alpha}(1-R)^{\beta}+b\{M_0+M_1(1-R)^{\gamma}+w(1-R)\}-h_v\{a+b(n-1)T\}}{2b\{h_v-2bk_1k_2p^{-\alpha}(1-R)^{-\beta}\}}$$

This proves the proposition.

Proposition 4 *For known t_m, the profit function $\Pi_v(t_m, R)$ is concave with respect to R for all $R > 1 - \min\{X_6, X_7\}$ where, $X_6 = \frac{(1+\beta)M_0}{(1-\beta)w}$ and* $X_7 = [\frac{2a^2k_1k_2(2\beta+1)(T-t_m)p^{-\alpha}}{WT(1+\beta)\{a+\frac{b}{2}(2n+1)T\}}]^{\frac{1}{\beta}}$.

Proof. Differentiating (5) twice with respect to R, we get

$$\frac{d^2\Pi_v(t_m,R)}{dR^2} = -\frac{p^{-2\alpha}(1-R)^{-2\beta}}{6T}[6\{2a^2k_1k_2\beta(2\beta+1)(T-t_m) - \beta(\beta+1)WT\{a + \frac{b}{2}(2n+1)T\}p^{\alpha}(1-R)^{\beta}\} + 4bk_1^2k_2(T-t_m)\beta(2\beta+1)\{3a\{(2n + 1)T+t_m\} + b\{(3n^2+3n+1)T^2+(3n+1)Tt_m+t_m^2\}\} + 3k_1(T - t_m)p^{\alpha}(1-R)^{\beta}[2\beta\{M_0(1+\beta) - w(1-\beta)(1-R)\}\{a+\frac{b}{2}((2n + 1)T+t_m)\} +M_1(2a+b)(\beta-\gamma)(1+\beta-w)(1-R)^{\gamma} + \beta(\beta+1)(T-t_m) \{a+\frac{b}{3}((3n+1)T+2t_m)\}h_v]]$$

Clearly, the profit function $\Pi_v(t_m, R)$ will be concave with respect to R if the following two conditions are satisfied:

$(i)\ 2a^2k_1k_2\beta(2\beta+1)(T-t_m)-\beta(\beta+1)WT\{a+\frac{b}{2}(2n+1)T\}p^{\alpha}(1-R)^{\beta}>0$

$(ii)\ M_0(1+\beta)-w(1-\beta)(1-R)>0$

From (i) we have $(1-R)<[\frac{2a^2k_1k_2(2\beta+1)(T-t_m)p^{-\alpha}}{WT(1+\beta)\{a+\frac{b}{2}(2n+1)T\}}]^{\frac{1}{\beta}}=X_7$

From (ii) we have $(1-R)<\frac{(1+\beta)M_0}{(1-\beta)w}=X_6$

Hence, the proposition is proved.

4.2. CENTRALISED MODEL

The average total profit of the integrated supply chain is given by

$$\Pi(T,t_m,p,R)=\Pi_v(t_m,R)+\Pi_b(T,p)$$

$$=[p\{a+\frac{b}{2}(2n+1)T\}-\frac{Th_b}{2}\{a+\frac{b}{3}(3n+2)T\}]p^{-\alpha}(1-R)^{-\beta}-\frac{F+S_b+S_v}{T}$$

$$-\frac{k_1h_b(T-t_m)}{4z}[4a^2+2ab\{2(2n+1)T+t_m\}+b^2\{(2n+1)T+t_m\}]p^{-2\alpha}$$

$$(1-R)^{-2\beta}-\frac{k_1(T-t_m)}{T}[\{d+w(1-R)-(T-t_m)(M_0+M_1(1-R)^{\gamma})\}\{a+$$

$$\frac{b}{2}((2n+1)T+t_m)\}-\frac{h_v(T-t_m)}{2}\{a+\frac{b}{3}((3n+1)T+2t_m)\}]p^{-\alpha}(1-R)^{-\beta}$$

$$-\frac{k_1^2(T-t_m)^2}{T}[(k_2+\frac{h_b}{z})\{a^2+ab\{(2n+1)T+t_m\}+n(n+1)b^2T^2\}$$

$$+\frac{b^2h_b}{4z}(T^2-t_m^2)+\frac{k_2b^2}{3}\{T^2+(3n+1)Tt_m+t_m^2\}]p^{-2\alpha}(1-R)^{-2\beta} \tag{11}$$

Proposition 5 *In case of the centralized supply chain system, the product reliability R depends on the decision variables T and t_m given by the relation*

$$R(T,t_m)=\frac{\{a+b(n+\frac{1}{2})T\}T}{k_1(T-t_m)[a+\frac{b}{2}\{(2n+1)T+t_m\}]} \tag{12}$$

Proof. The vendor delivers Q_n quantity of items to the buyer, of which $(1-R)\%$ is found to be defective after completion of the buyer's screening process. Hence, only Q_nR quantity is considered as good items and sold by the buyer to meet the market demand D_n. Since, there is no shortage and no excess items, we can claim that $Q_nR=D_n$. Using (1) and (2), we have

$$R(T,t_m)=\frac{\{a+b(n+\frac{1}{2})T\}T}{k_1(T-t_m)[a+\frac{b}{2}\{(2n+1)T+t_m\}]}$$

Proposition 6 *The buyer's selling price p depends on the decision variables T and t_m given by the relation* $p(T,t_m)=(1-R)^{\frac{-\beta}{\alpha}}[1+\frac{bT\{a+\frac{b}{2}(2n+1)T\}}{2z^2R^2}]^{\frac{1}{2\alpha}}$ (13)

where $R = R(T, t_m)$ is given by (12).

Proof. Substituting the value of Q_n from (2) into the relation (9), we get

$$T\{a + \frac{b}{2}(2n+1)T = k_1(T - t_m)p^{-2\alpha}(1-R)^{-2\beta}[a + \frac{b}{2}\{(2n+1)T + t_m\}][R + \frac{b}{2z^2}k_1 (T - t_m)[a + \frac{b}{2}\{(2n+1)T + t_m\}]]$$

$$\text{or, } p^{2\alpha}(1-R)^{2\beta} = \frac{k_1(T - t_m)\left[a + \frac{b}{2}\{(2n+1)T + t_m\}\right]}{\left\{a + b\left(n + \frac{1}{2}\right)T\right\}T}[R + \frac{b}{2z^2}k_1(T - t_m) [a + \frac{b}{2}\{(2n+1)T + t_m\}]]$$

$$\text{or, } p = (1-R)^{\frac{-\beta}{\alpha}}[1 + \frac{bT\{a + \frac{b}{2}(2n+1)T\}}{2z^2R^2}]^{\frac{1}{2\alpha}} \quad [\text{using}(12)]$$

Hence, the proposition is proved.

Proposition 7 *To meet the customer demand D_n, the vendor produces Q_n quantity of items with delay in time t_m satisfying the relation*

$$0 < t_m < \frac{\sqrt{k_1^2(a + bnT)^2 + bk_1(k_1 - 1)T\{2a + b(2n+1)T\}} - k_1(a + bnT)}{bk_1}.$$

Proof. Since $0 < R(T, t_m) < 1$, therefore, from (12) it is obvious that $R(T, t_m) > 0$.

Again, $R(T, t_m) < 1$ gives$\{a + \frac{b}{2}(2n+1)T\}T < k_1(T - t_m)[a + \frac{b}{2}\{(2n+1)T + t_m\}]$

or, $bk_1t_m^2 + 2k_1(a + bnT)t_m - 2(k_1 - 1)T\{a + \frac{b}{2}(2n+1)T\} < 0$

Considering the above inequation as equation, we see that the two roots of the equation are

$$t_m = \frac{\pm\sqrt{k_1^2(a + bnT)^2 + bk_1(k_1 - 1)T\{2a + b(2n+1)T\}} - k_1(a + bnT)}{bk_1}$$

The smaller root is negative and hence the proposition is proved.

Using (12) and (13), the profit function $\Pi(T, t_m, p, R)$ can be reduced to the function $\Pi(T, t_m)$ of two independent variables T and t_m. It is not possible to prove analytically that $\Pi(T, t_m)$ is jointly concave. However, we can prove the following proposition:

Proposition 8 *For known values of* p, R *and* T, *the profit function* $\Pi(T, t_m)$ *is concave with respect to* t_m *for all* $t_m \gtrless \frac{-[X_8T^2+X_9T-2X_{10}]}{4b\{h_vp^{\alpha}(1-R)^{\beta}-bk_1k_2\}}$ *according as* $h_v \gtrless bk_1k_2p^{-\alpha}(1-R)^{-\beta}$ *and* T *satisfies the relation* $X_8T^2 + X_9T - 2X_{10} \lesseqgtr 0$ *where,*

$$X_8 = \frac{b^2h_b(2n+1)}{z}$$

$$X_9 = \{\frac{ab}{z} + 4Rp^{2\alpha}(1-R)^{2\beta}\}h_b + 2b(n-1)h_vp^{\alpha}(1-R)^{\beta} - 4b^2nk_1k_2$$

$$X_{10} = b\{M_0 + M_1(1-R)^{\gamma} + d + w(1-R)\}p^{\alpha}(1-R)^{\beta} + 2abk_1k_2 - ah_vp^{\alpha}(1-R)^{\beta}$$

Proof. Differentiating (11) twice with respect to t_m, we get

$$\frac{\partial^2\Pi}{\partial t_m^2} = -\frac{k_1p^{-2\alpha}(1-R)^{-2\beta}}{2T}[4b\{h_vp^{\alpha}(1-R)^{\beta} - bk_1k_2\}t_m + \{X_8T^2 + X_9T - 2X_{10}\}]$$

The profit function $\Pi(T, t_m)$ will be concave with respect to t_m if

$$4b\{h_vp^{\alpha}(1-R)^{\beta} - bk_1k_2\}t_m + \{X_8T^2 + X_9T - 2X_{10}\} > 0.$$

For $h_v \gtrless bk_1k_2p^{-\alpha}(1-R)^{-\beta}$, we have $t_m \gtrless \frac{-[X_8T^2+X_9T-2X_{10}]}{4b\{h_vp^{\alpha}(1-R)^{\beta}-bk_1k_2\}}$.

Since the vendor's production delay time (t_m) is always positive, the numerator of the right hand expression must be positive and hence $X_8T^2 + X_9T - 2X_{10} \lesseqgtr 0$. This proves the proposition.

Proposition 9 *For pre-defined values of* p, R *and* t_m, *the profit function* $\Pi(T, t_m)$ *is concave with respect to* T *if*

$$[\frac{2k_1k_2z(3a^2 + 3abt_m + b^2t_m^2)}{3n(2n+1)b^2h_b}]^{\frac{1}{3}} < T < \frac{1}{b(2n+1)}[\frac{4bk_1k_2z(3n^2+3n+1)}{9(2n+1)h_b} - a]$$

$$\text{and } \frac{b}{2R}\{d + w(1-R) + M_0 + M_1(1-R)^{\gamma}\}p^{-\alpha}(1-R)^{-\beta} < h_b < \frac{4bk_1k_2z(3n^2+3n+1)}{9(2n+1)a}$$

Proof. Differentiating (11) twice with respect to T, we have

$$\frac{d^2\Pi}{dT^2} = -\frac{p^{-2\alpha}(1-R)^{-2\beta}}{6bT^3z}[6k_1p^{\alpha}(1-R)^{\beta}t_mz(2a+bt_m)\{2h_bp^{\alpha}(1-R)^{\beta}R$$
$$-b\{d + w(1-R) + M_0 + M_1(1-R)^{\gamma}\}\}$$
$$+2(3n+2)b^2zh_bp^{\alpha}(1-R)^{\beta}T^3 + 12bzp^{2\alpha}(1-R)^{2\beta}(F + S_b + S_v)$$
$$+2bzk_1h_vp^{\alpha}(1-R)^{\beta}\{(3a+2bt_m)t_m^2 + b(3n+1)T^3\}$$
$$+b^2k_1T^3\{4bk_1k_2z(3n^2+3n+1) - 9(2n+1)\{a + b(2n+1)T\}h_b\}$$
$$+2bk_1t_m\{3b^2n(2n+1)h_bT^3 - 2k_1k_2z(3a^2 + 3abt_m + b^2t_m^2)\}]$$

In the right-hand side of the above equation, the expression within the third bracket will be positive if the following three conditions are satisfied:

$$2h_b p^{\alpha}(1-R)^{\beta}R - b\{d + w(1-R) + M_0 + M_1(1-R)^{\gamma}\} > 0$$

$$4bk_1k_2z(3n^2+3n+1) - 9(2n+1)\{a + b(2n+1)T\}h_b > 0$$

$$3b^2n(2n+1)h_bT^3 - 2k_1k_2z(3a^2 + 3abt_m + b^2t_m^2) > 0$$

From (i) we have, $h_b > \frac{b}{2R}\{d + w(1-R) + M_0 + M_1(1-R)^{\gamma}\}p^{-\alpha}(1-R)^{-\beta}$.

From (ii) we have, $T < \frac{1}{b(2n+1)}\left[\frac{4bk_1k_2z(3n^2+3n+1)}{9(2n+1)h_b} - a\right]$ provided that $h_b < \frac{4bk_1k_2z(3n^2+3n+1)}{9(2n+1)a}$

From (iii) we have $T^3 > \frac{2k_1k_2z(3a^2+3abt_m+b^2t_m^2)}{3n(2n+1)b^2h_b}$.

Hence, the proposition is proved.

5. NUMERICAL EXAMPLE

To illustrate the developed models numerically, we consider the following data-set (Giri and Maiti (2012)):

$a = 200; b = 2; S_b = 50; S_v = 80; h_b = 0.2; h_v = 0.08$ and $k_1 = 1.5$. Also, we consider $k_2 = 0.015; w = 2; W = 10; n = 6, ; F = 25; z = 100; d = 0.02$ in appropriate units.

To check the concavity of the profit function $\Pi(T, t_m, p, R)$, we observe that α, β, M_0 and M_1 have to satisfy the conditions$0 \le \alpha \le 0.16;\ \beta \ge 0.18;\ 0 \le M_0 \le 1.1;\ 0 \le M_1 \le 1$and the variable γ has no restriction. So, we consider $\alpha = 0.15; \beta = 0.2; \gamma = 0.2; M_0 = 0.4$and $M_1 = 0.01$. The decision variablesR^* and $p^*$are found from the propositions 6 and 7, respectively as $R^* = 0.837063$ and $p^* = 13.0073$. Then we have, $\frac{\partial^2\Pi}{\partial T^2} = -543.54 < 0, \frac{\partial^2\Pi}{\partial t_m^2} = -532.319 < 0$ and the determinant of Hessian matrix associate with $\Pi(T, t_m)$ is given by

$$\begin{vmatrix} \frac{\partial^2\Pi}{\partial T^2} & \frac{\partial^2\Pi}{\partial T\partial t_m} \\ \frac{\partial^2\Pi}{\partial t_m\partial T} & \frac{\partial^2\Pi}{\partial t_m^2} \end{vmatrix} = \begin{vmatrix} -543.54 & -503.414 \\ -503.414 & -532.319 \end{vmatrix} = 35911.3 > 0$$

This proves that, for the above data set, the profit function $\Pi(T, t_m, p, R)$ is concave in T and t_m. One evidence is shown in Figure 2 for $n = 6$. We observe that, if we move from one cycle to the next cycle, the buyer's ordering time period (T) and the vendor's delay time (t_m) to start production change very slowly whereas the average total profit of the supply chain increases considerably. Without any loss of generality, we consider the sixth cycle $(n = 6)$ and we obtain $T^* = 0.741639$ and $t_m^* = 0.151398$. In this sixth cycle, the vendor produces $Q_n^* = 187.99$ quantity of items. After receiving these items, the buyer performs screening and 83.7% of Q_n^* $i.e.$, 157.36 quantity of items is considered as good quality and perfect items to meet the demand D_n given by (1).

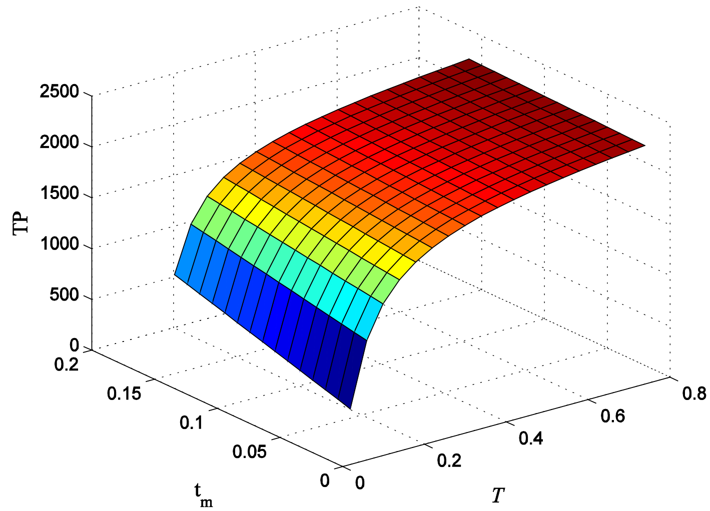


Figure 2 Graphical representation of the profit function $\Pi(T, t_m)$ for $n = 6$

The buyer's selling price p, the product reliability R and the average total profit of the supply chain increase as we move from one cycle to the next cycle. Since the changes inT^*andt_m^* are insensitive, we present in Table 1 the values of p^*, R^* and $\Pi(T^*, t_m^*, p^*, R^*)$ for successive ten cycles.

Table 1 Optimal results of the proposed model for successive ten cycles

n-th cycle	p^*	R^*	$\Pi(T^*, t_m^*, p^*, R^*)$
1	12.9386	0.874390	2035.33
2	12.9523	0.837046	2053.32
3	12.9660	0.837050	2071.32
4	12.9798	0.837055	2089.34
5	12.9935	0.837059	2107.37
6	13.0073	0.837063	2125.42
7	13.0210	0.837068	2143.48
8	13.0348	0.837072	2161.56
9	13.0485	0.837076	2179.66
10	13.0623	0.837080	2197.77

5.1. THE CASE OF $\alpha = 0, \beta = 0$

In this scenario, we assume that the demand rate depends on time only and hence we put $\alpha = 0$ and $\beta = 0$ in our proposed model. The demand rate becomes $D(t) = a + bt$ and the vendor's production rate is $P(t) = k_1 D(t)$, with $k_1 > 0$. Also, we assume that unit production cost does not depend on reliability and it is fixed and denoted by C_p. To compare the results with the optimal results of our proposed model, we take $C_p = 7.1253, p = 13.0073$ and $R = 0.837063$. This

Table 2 Comparison of the results of our model and the model with $\alpha = 0, \beta = 0$

	Model with $\alpha = 0, \beta = 0$			**Our model**	**Difference of profits**
n-th cycle	T^*	t_m^*	$\Pi^a(T^*, t_m^*)$	$\Pi(T^*, t_m^*, p^*, R^*)$	$\Pi - \Pi^a$
1	3.19089	1.06363	1016.77	2035.33	1018.56
2	4.10970	1.36990	1052.09	2053.32	1001.23
3	4.91770	1.63923	1092.64	2071.32	978.68
4	5.32066	1.77355	1134.25	2089.34	955.09
5	5.44683	1.81561	1174.16	2107.37	933.21
6	5.43545	1.81182	1211.56	2125.42	913.86
7	5.35954	1.78651	1246.45	2143.48	897.03
8	5.25452	1.75151	1279.06	2161.56	882.50
9	5.13785	1.71262	1309.67	2179.60	869.93
10	5.01832	1.67277	1338.52	2197.77	859.25

implies that 83.7% of received items from the vendor is sold by the buyer at the retail price \$13.0 to meet the market demand. All the remaining assumptions are kept unchanged. Thus, we take $\alpha = 0, \beta = 0, M_0 = 0, M_1 = 0, k_2 = 0$, $\gamma = 0$ and $C(t, p, R) = C_p = 7.1253$ and all other parameter-values are same as assumed before. With this data-set, we find that for $n = 6, T^* = 5.43545, t_m^* = 1.81182$ and the average total profit of the supply chain as \$1211.56 which is \$2125.42 less than that of our proposed model. In Table 2, we compare the optimal results of ten successive cycles with those of the proposed model.

5.2. THE CASE OF $\mathbf{R = 1}$

Here we assume that the vendor's produced items are all perfect, although in reality it may not always happen. To compare the results with those of the proposed model, we assume the market

Table 3 Comparison of profits of the proposed model and our model with $R = 1$

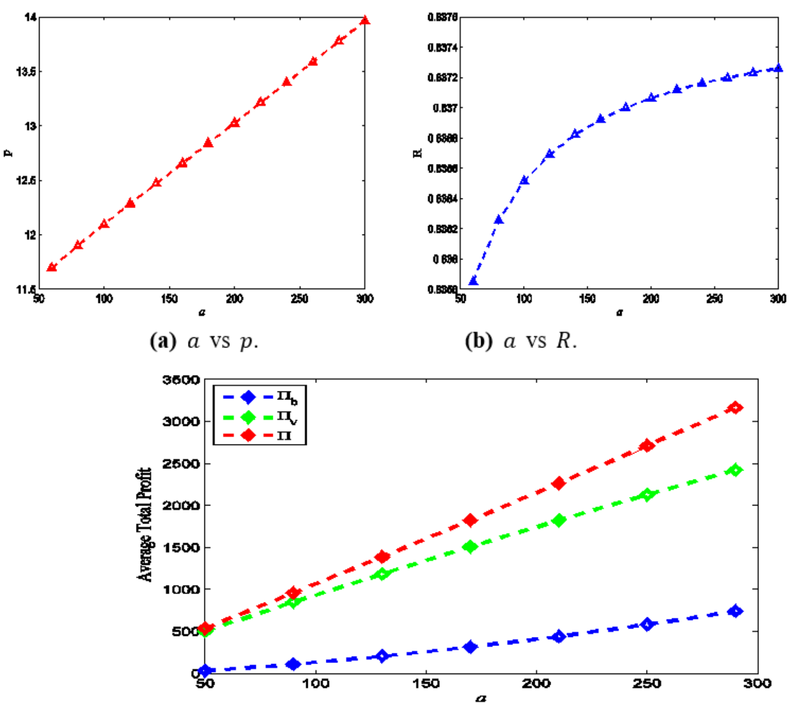
	Model with $R = 1$			**Our model**	**Difference of profits**
n-th cycle	T^*	t_m^*	$\Pi^b(T^*, t_m^*)$	$\Pi(T^*, t_m^*, p^*, R^*)$	$\Pi - \Pi^b$
1	4.10977	1.36992	729.98	2035.33	1305.35
2	5.99780	1.99927	765.33	2053.32	1287.99
3	8.86478	2.95493	815.01	2071.32	1256.31
4	11.4611	3.82038	878.73	2089.34	1210.61
5	13.3829	4.46097	951.72	2107.37	1155.65
6	14.7948	4.93159	1030.37	2125.42	1095.05
7	15.8619	5.28731	1112.63	2143.48	1030.85
8	16.6930	5.56433	1197.30	2161.56	964.26
9	17.3571	5.78570	1283.65	2179.60	895.95
10	17.8994	5.96648	1371.22	2197.77	826.55

demand as $D(t) = (a + bt)p^{\alpha}$, where $p = 13.0$ and $\alpha = 0.015$. The production rate is $P(t) = k_1 D(t)$, with $k_1 > 0$. In this case, the buyer's holding cost changes to $h_b T[(n+1)\{a + \frac{b}{2}(n+1)T - (2n+1)\frac{a}{2} - (3n^2 + 3n + 1)\frac{bT}{6}\}]p^{-\alpha}$. As before, we assume that unit production cost $C_p = 7.13$. Since, all products are perfect, there is no need to screen and hence we take $w = 0, d = 0$ and $z = 0$. All the remaining assumptions are kept unchanged. Thus, in numerical data, we take $R = 1, M_0 = 0, M_1 = 0, k_2 = 0$, $\gamma = 0$ and $C(t, p, R) = C_p = 7.13$, keeping all other parameter-values unchanged. From the numerical experiment, we find that $n^* = 6$, $T^* = 14.7948$, $t_m^* = 4.93159$ and the average total profit of the supply chain model is \$1030.37, which is \$2125.42 less than that of our proposed model. In Table 3, we compare the optimal results of ten successive cycles with those of our proposed model.

6. SENSITIVITY ANALYSIS

In this section, we investigate the effect of change of one parameter-value at a time keeping the remaining parameter-values unchanged. The sensitivity of the parameters a, b, α, β and k_1 are shown in the Figure 3, Figure 4, Figure 5, Figure 6, Figure 7. Some insights from our investigation are given below.

1) Both the buyer's selling price (p) and the product reliability (R) increase rapidly as a increases (Figure 3(a, b)). The vendor has to produce more reliable product as a increases. As a result, the vendor's unit production cost increases and at the same time, the market



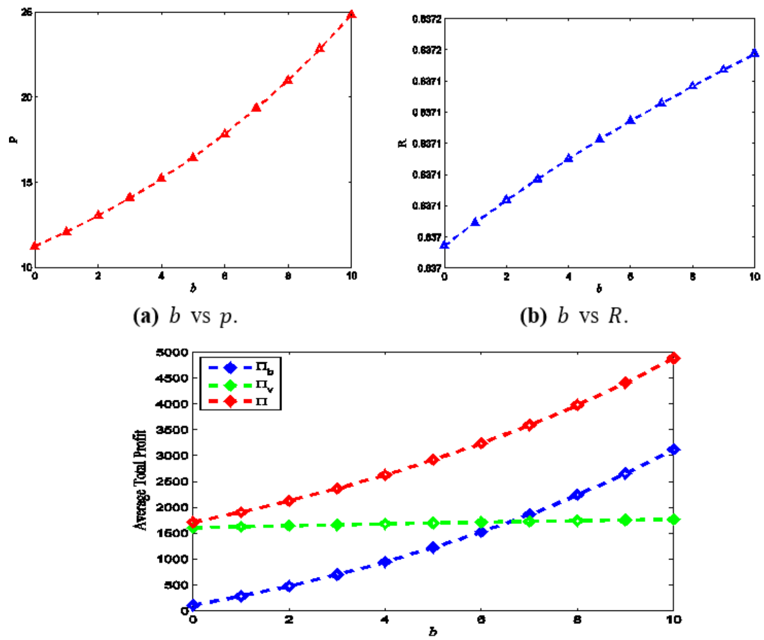
(a) a vs p. **(b)** a vs R.

(c) a vs average total profit.

Figure 3 Change (%) in optimal results w.r.t. a.

demand also increases. Therefore, the buyer's average total profit as well as the vendor's average total profit increase as a increases. Consequently the average total profit of the integrated supply chain increases as a increases (Figure 3 (c)).

2) As b increases, the selling price p increases but the rate of increase in R is not so high. The buyer's average total profit increases significantly but the vendor's average total profit increase is very low. As a result, the average total profit of the integrated supply chain model increases moderately as the value of b increases (Figure 4 (a, b, c)).

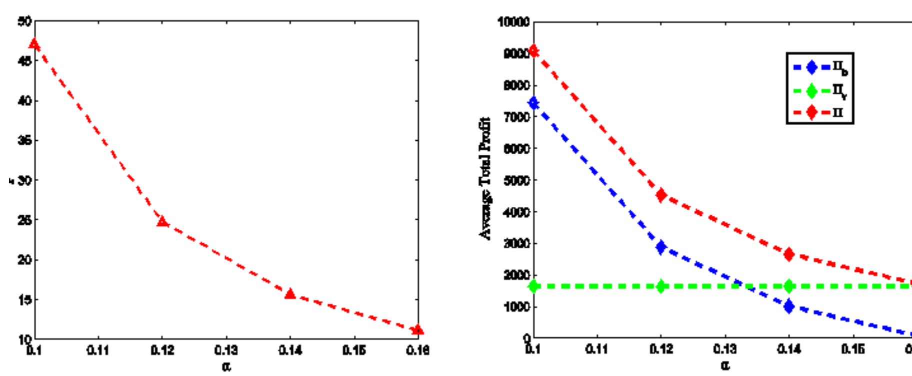


(a) b vs p. **(b)** b vs R.

(c) b vs average total profit.

Figure 4 Change (%) in optimal results w.r.t. b.

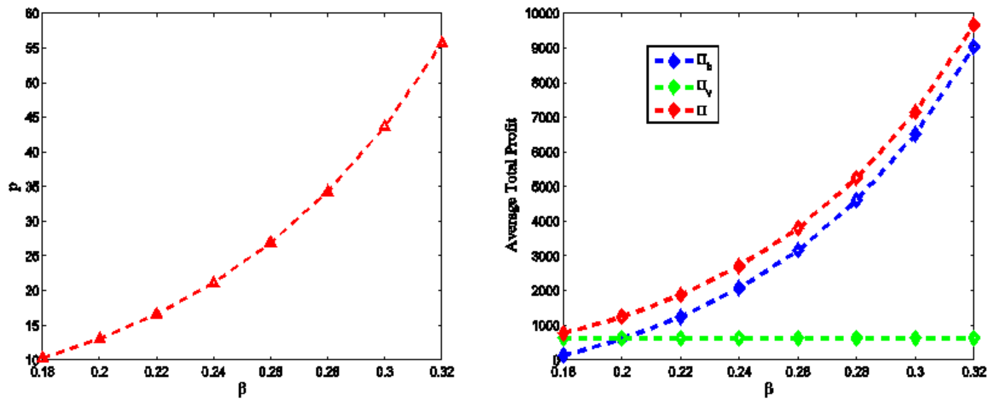
3) The product reliability (R) is not affected by the price elasticity to demand (α) but the buyer's selling price is highly sensitive with respect to α as shown in Figure 5(a). A 10% increase in the value of α results 83% decrease in the value of the selling price p. But it does not have any impact on the vendor's average total profit. A lower selling price results in lower profit from the buyer's perspective as well as from the integrated supply chain's perspective (Figure 5 (b)).



(a) α vs p. **(b)** α vs average total profit.

Figure 5 Change (%) in optimal results w.r.t. α.

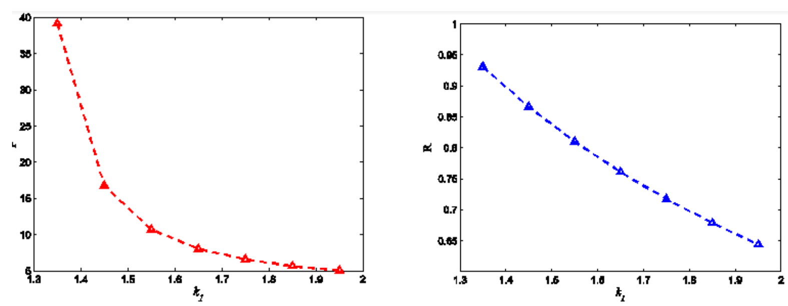
4) As β increases, the selling price p and the average total profits of the buyer and the entire supply chain increase (Figure $6(a, b)$).



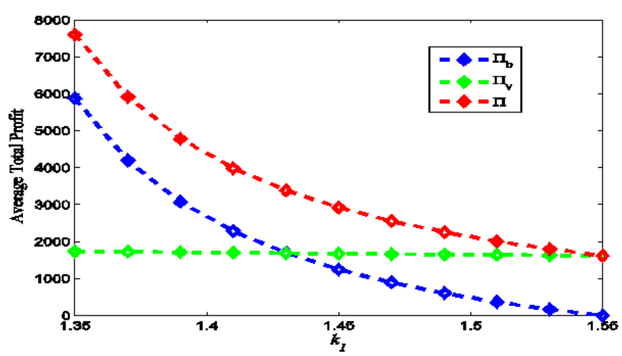
(a) β vs p. **(b)** β vs average total profit.

Figure 6 Change (%) in optimal results w.r.t. β.

5) Figure $7(a)$ shows that, as k_1 increases, the buyer's selling price and reliability of the product decrease (Figure $7(b)$). Due to increase in production rate, the vendor's production time decreases but there is at most no change in the average total profit of the vendor. However, the average total profit of integrated supply chain decreases as k_1 increases (Figure $7(c)$).



(a) k_1 vs p. **(b)** k_1 vs R.



(c) k_1 vs average total profit.

Figure 7 Change (%) in optimal results w.r.t. k_1.

7. CONCLUSION

The paper considers a single vendor single buyer integrated supply chain model in which the market demand is assumed to be dependent on time, price and reliability of the product. The vendor follows a lot-for-lot policy. The items are delivered to the buyer with an agreement that the buyer himself screens all those products and, if any item is found defective, it should be sold with price discount and the cost must be borne by the vendor. The reputation of the vendor and the buyer increase as the product bears good and perfect quality to the best of their knowledge. On the other hand, the end customer's satisfaction increases as the product is more reliable. In this paper, some propositions are derived which help to choose the data-set in the numerical example as well as to find the optimal values of the decision variables. From the numerical analysis, we have found that the vendor has to maintain the reliability of the product and produce items not more 13% defective. It is also observed that the scaling constant α, β for the demand act important roles to increase the profits of the buyer, vendor and the integrated supply chain.

In this article, we have assumed a deterministic market demand, which has limited applications in the business world. So, this model can be extended by considering stochastic demand. Shortages are not allowed in our model. So, one can extend the present model with inclusion of shortage in the buyer's inventory. One can also consider multi-vendor and/or multi-buyer supply chain for further study. Terms and conditions may be imposed by the vendor to sell the defective items (from buyer's screening) with price discount.

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