ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION

6z^2 = 6x^2 – 5y^2

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Abstract:
The ternary homogeneous quadratic equation given by 6z^2 = 6x^2 – 5y^2 representing a cone is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramided numbers are presented. Also, given a solution, formulas for generating a sequence of solutions based on the given solutions are presented.

Keywords:
Ternary quadratic, integer solutions, figurate numbers, homogeneous quadratic, polygonal number, pyramidal numbers.

Notations Used:
1. Polygonal number of rank ‘n’ with sides m

\[ t_{m,n} = n \left(1 + \frac{(n-1)(m-2)}{2}\right) \quad m=3, 4, 5..., 10... \]

2. Stella octangular number of rank ‘n’

\[ SO_n = n(2n^2 - 1) \]


1. INTRODUCTION

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-24] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation 6z^2 = 6x^2 - 5y^2 representing non-homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.
2. METHOD OF ANALYSIS

The ternary quadratic Diophantine equation representing a cone under consideration is

\[ 6z^2 = 6x^2 - 5y^2 \]  \hspace{1cm} (1)

To start with, it is seen that (1) is satisfied by the following non-zero integer triples \((x, y, z)\):

\((1921,2088,239); (466,504,74); (603,648,117); (439,468,101);

\((341,360,91); (138,144,42); (169,168,71); (74,72,34); (87,72,57); (34,24,26).\)

As the considered equation is symmetric in \(x, y\) and \(z\), we have presented only positive integer solutions for clear understanding.

However, we have other choices of solution to (1) which are illustrated below:

**Choice 1:** Introducing the linear transformations

\[ x = 6X + 5\alpha - 30\beta \]
\[ y = 6X + 6\alpha - 36\beta \]
\[ z = \alpha + 30\beta \]  \hspace{1cm} (2)

In (1), it is written as

\[ X^2 = \alpha^2 + 180\beta^2 \]  \hspace{1cm} (3)

Which is satisfied by

\[ \beta = 2pq \]
\[ \alpha = 180p^2 - q^2 \]  \hspace{1cm} (4)
\[ X = 180p^2 + q^2 \]

In view of (2), the non-zero distinct integer solutions to (1) are given by

\[ x(p, q) = 1980p^2 + q^2 - 60pq \]
\[ y(p, q) = 2160p^2 - 72pq \]
\[ z(p, q) = 180 - q^2 + 60pq \]  \hspace{1cm} (5)

**Properties:**

A few interesting properties obtained are as follows:

\[ x(p, p + 1) + y(p, p + 1) - 4140t_{4,p} + 66(2p^2 - p) - 2t_{3,p} \equiv 1 \pmod{197} \]

\[ x(p + 1, q) + z(p + 1, q) - 4320t_{3,p} \equiv 1 \pmod{2160} \]
Choice 2:

(3) is written as the system of double equations as given below

System 1 $\rightarrow X + \alpha = \beta^2$
\[X - \alpha = 180\]

System 2 $\rightarrow X + \alpha = 2\beta^2$
\[X - \alpha = 90\]

System 3 $\rightarrow X + \alpha = 3\beta^2$
\[X - \alpha = 60\]

System 4 $\rightarrow X + \alpha = 5\beta^2$
\[X - \alpha = 36\]

System 5 $\rightarrow X + \alpha = 6\beta^2$
\[X - \alpha = 30\]

System 6 $\rightarrow X + \alpha = 9\beta^2$
\[X - \alpha = 20\]

System 7 $\rightarrow X + \alpha = 10\beta^2$
\[X - \alpha = 18\]

System 8 $\rightarrow X + \alpha = 15\beta^2$
\[X - \alpha = 12\]

System 9 $\rightarrow X + \alpha = 18\beta^2$
\[X - \alpha = 10\]

Solving each of the above systems, the corresponding values of $X$, $\alpha$ and $\beta$ are determined. Substituting these values of $X$, $\alpha$ and $\beta$ in (2) the corresponding integer solutions to (1) are obtained which are exhibited in the table below.
Table: Solutions of (1)

<table>
<thead>
<tr>
<th>SYSTEM</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$22k^2 - 60k + 90$</td>
<td>$24k^2 - 72k$</td>
<td>$2k^2 + 60k - 90$</td>
</tr>
<tr>
<td>2</td>
<td>$11\beta^2 - 30\beta + 45$</td>
<td>$12\beta^2 - 36\beta$</td>
<td>$\beta^2 + 30\beta - 45$</td>
</tr>
<tr>
<td>3</td>
<td>$66k^2 - 60k + 30$</td>
<td>$72k^2 - 72k$</td>
<td>$6k^2 + 60k - 30$</td>
</tr>
<tr>
<td>4</td>
<td>$110k^2 - 60k + 18$</td>
<td>$120k^2 - 72k$</td>
<td>$10k^2 + 60k - 18$</td>
</tr>
<tr>
<td>5</td>
<td>$33\beta^2 - 30\beta + 15$</td>
<td>$36\beta^2 - 36\beta$</td>
<td>$\beta^2 + 30\beta - 45$</td>
</tr>
<tr>
<td>6</td>
<td>$198k^2 - 60k + 10$</td>
<td>$216k^2 - 72k$</td>
<td>$18k^2 + 60k - 10$</td>
</tr>
<tr>
<td>7</td>
<td>$55\beta^2 - 30\beta + 9$</td>
<td>$60\beta^2 - 36\beta$</td>
<td>$5\beta^2 + 30\beta - 9$</td>
</tr>
<tr>
<td>8</td>
<td>$330k^2 - 60k + 6$</td>
<td>$360k^2 - 72k$</td>
<td>$30k^2 + 60k - 6$</td>
</tr>
<tr>
<td>9</td>
<td>$99\beta^2 - 30\beta + 5$</td>
<td>$108\beta^2 - 36\beta$</td>
<td>$9\beta^2 + 30\beta - 5$</td>
</tr>
</tbody>
</table>

Choice 3:

In (3), it is written as

$$\alpha^2 + 180\beta^2 = X^2 = X^2 \times 1$$  \hspace{1cm} (6)

Assume

$$X(a,b) = a^2 + 180b^2$$  \hspace{1cm} (7)

Write 1 as

$$1 = \frac{(2 + i\sqrt{5})(2 - i\sqrt{5})}{3^2}$$  \hspace{1cm} (8)

Substituting (7) & (8) in (6) and applying the method of factorization define

$$(\alpha + i6\sqrt{5}\beta) = (a + i6\sqrt{5}b)^2 \times \frac{(2 + i\sqrt{5})}{3}$$

Equating the real and imaginary parts and replacing a by 6A and b by 6B, we have

$$\alpha = 24A^2 - 720AB - 4320B^2$$  \hspace{1cm} (9)

$$\beta = 2A^2 + 48AB - 360B^2$$  \hspace{1cm} (10)

and from (7),  \hspace{1cm} $X = 36A^2 + 6480B^2$

Substituting the above values of $\alpha$, $\beta$, $X$ in (2), the corresponding integer solutions to (1) are given by
\[ x(A, B) = 276A^2 - 5040AB + 28080B^2 \]
\[ y(A, B) = 288A^2 - 6048AB + 25920B^2 \]  \hspace{1cm} (11)
\[ z(A, B) = 84A^2 + 720AB - 15120B^2 \]

**Properties:**

A few interesting properties obtained are as follows:

- \( x(A + 1, A) - y(A + 1, A) - 789t_{10, A} \equiv 996 \pmod{2343} \)
- \( x(A - 1, A) - y(A - 1, A) - z(A - 1, A) - 5824t_{8, A} \equiv 96 \pmod{11552} \)
- \( y(A + 1, A) - z(A + 1, A) - 30660t_{4, A} + 2544t_{7, A} - 204 = 0 \)
- \( z(A, A) = -14316t_{4, A} \)
- \( x(A, A) + z(A, A) - 432t_{6, A} + 77760t_{4, A} \equiv 0 \pmod{432} \)

Note that instead of (8), one may also write 1 as

\[ 1 = \frac{(-2 + i\sqrt{5})(-2 - i\sqrt{5})}{3^2} \]

For this choice, the corresponding integer solutions to (1) are given by

\[ x(A, B) = 36A^2 - 2160AB + 71280B^2 \]
\[ y(A, B) = -2592AB + 77760B^2 \]
\[ z(A, B) = 36A^2 - 2160AB - 6480B^2 \]

### 3. REMARKABLE OBSERVATIONS

Let \((x_0, y_0, z_0)\) be any given integer solution to (1). We illustrate below the method of obtaining a general formula for generating sequence of integer solutions based on the given solution.
Case (i):

Let 
\[ x_1 = -2x_0 + h \]
\[ y_1 = 2y_0 + h \]
\[ z_1 = 2z_0 \]

be the second solution of \((1)\). Substituting \((12)\) in \((1)\) & performing a few calculations, we have

\[ h = 20y_0 + 24x_0 \text{ and then} \]
\[ x_1 = 22x_0 + 22y_0 \]
\[ y_1 = 22y_0 + 24x_0 \]

This is written in the form of matrix as

\[
\begin{pmatrix}
  x_1 \\
  y_1 
\end{pmatrix} = M \begin{pmatrix}
  x_0 \\
  y_0 
\end{pmatrix}
\]

(13)

where 
\[
M = \begin{pmatrix}
  24 & 20 \\
  24 & 22 
\end{pmatrix}
\]

Repeating the above process, the general solution \((x_n, y_n)\) to \((1)\) is given by

\[
\begin{pmatrix}
  x_n \\
  y_n
\end{pmatrix} = M^n \begin{pmatrix}
  x_0 \\
  y_0
\end{pmatrix}
\]

(14)

To find \(M^n\), the eigen values of \(M\) are \(\alpha = 1, \beta = 1\).

We know that

\[
M^n = \frac{\alpha^n}{\alpha - \beta} (M - \beta I) + \frac{\beta^n}{\beta - \alpha} (M - \alpha I)
\]

Using the above formula, we have

\[
M^n = \frac{1}{4\sqrt{480}} \begin{pmatrix}
  2\sqrt{480}(A^n) & 20(B^n) \\
  24(B^n) & 2\sqrt{480}(A^n)
\end{pmatrix}
\]

Thus the general solution \((x_n, y_n, z_n)\) to \((1)\) is given by

\[
x_n = \frac{1}{4\sqrt{480}} \left[ (2\sqrt{480}A^n)x_0 + 20B^n) y_0 \right]
\]
\[ y_n = \frac{1}{4\sqrt{480}} \left( 24B^n \right) x_0 + (2\sqrt{480}A^n)y_0 \]

\[ z_n = 2^n z_0 \]

Where

\[ A^n = (22 + 2\sqrt{480})^n + (22 - 2\sqrt{480})^n \]

\[ B^n = (22 + 2\sqrt{480})^n - (22 - 2\sqrt{480})^n \]

**Case (ii):**

Let

\[ x_1 = 3x_0 + h \]
\[ y_1 = 3y_0, \quad h \neq 0 \]
\[ z_1 = 3z_0 + h \]

Repeating the process as in the case (i) the corresponding general solution \((x_n, y_n, z_n)\) to (1) is given by

\[ x_n = \frac{1}{6} \left[ (4(9)^n + 2(3^n)x_0 + (2(9)^n + 3^n)z_0 \right] \]

\[ y_n = 3^n y_0 \]

\[ z_n = \frac{1}{6} \left[ (-4(9)^n - 4(3^n)x_0 + (2(9)^n + 3^n)z_0 \right] \]

**Case (iii):**

Let

\[ x_1 = x_0 \]
\[ y_1 = y_0 - 2h, \quad h \neq 0 \]
\[ z_1 = z_0 + 2h \]

Repeating the procedure as presented in Case (i), the corresponding general solution \((x_n, y_n, z_n)\) to (1) is given by

\[ x_n = x_0 \]

\[ y_n = -\frac{1}{22} \left[ (10(21)^n - 12(1)^n)y_0 + (2(-21)^n - 2(1)^n)z_0 \right] \]

\[ z_n = -\frac{1}{22} \left[ (10(-21)^n - 10(1)^n)y_0 + (12(-21)^n + 10(1)^n)z_0 \right] \]
4. CONCLUSION

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by

\[ 6z^2 = 6x^2 - 5y^2 \]

As quadratic equations are rich in variety, one may search for their choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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6. REFERENCES


