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CONSTRUCTION OF IRRATIONAL GAUSSIAN DIOPHANTINE QUADRUPLES

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Abstract:

Given any two non-zero distinct irrational Gaussian integers such that their product added with either 1 or 4 is a perfect square, an irrational Gaussian Diophantine quadruple (a_0, a_1, a_2, a_3) such that the product of any two members of the set added with either 1 or 4 is a perfect square by employing the non-zero distinct integer solutions of the system of double Diophantine equations. The repetition of the above process leads to the generation of sequences of irrational Gaussian Diophantine quadruples with the given property.

Keywords:

Diophantine quadruple, irrational Diophantine quadruples, Gaussian Diophantine quadruples, Pell equations, Double Diophantine equations, integer solutions.

Mathematics subject classification number: *11D09, 11D99*

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1. INTRODUCTION

The problem of constructing the sets with property that the product of any two of its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus. A set of m distinct non-zero integers $\{a_1, a_2, \ldots\ldots, a_m\}$ is called a Diophantine m-tuple with property D(n) if $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$.

Many mathematicians analyzed the construction of different formulations of Diophantine triple and Diophantine quadruples with property D(n) for any arbitrary integer n and also for polynomials in n. Further various authors considered the constructions of the Davenport and Fibonacci numbers in [2-15].

A set $\{a_{1,}a_{2,}......., a_m\} \subset Z(i)-\{0\}$ is said to have this property D(z) if the product of its any two distinct elements increased by z is a square of a Gaussian integer. If the set $\{a_{1,}a_{2,}......., a_m\}$ is a complex Diophantine quadruple then the same is true for the set $\{-a_{1,} -a_{2,}.......,-a_m\}$ .Particularly in [16], the authors have analyzed the problem of the existence of the complex Diophantine quadruples. In this context one may refer [17-22].

In this communication, we construct sequences of irrational Gaussian Diophantine quadruples with properties D(1) and D(4).

(A number of the form $a+ib\sqrt{d}$ ($d \succ 0$, and square free) is read as irrational Gaussian number.)

2. METHOD OF ANALYSIS

I: FORMATION OF SEQUENCE OF IRRATIONAL GAUSSIAN DIOPHANTINE QUADRUPLES WITH THE PROPERTY D(1):

SEQUENCE: I

Let
$$a_0 = (3A-2p)+i\sqrt{d}(3B-2q)$$
$$a_1 = (3A-8p+5)+i\sqrt{d}(3B-8q)$$

be such that

$$a_0a_1+1 = \{[3A-5p+1]+i\sqrt{d}[3B-5q]\}^2 = r^2 \text{ (say).}$$

where $A = p^2-dq^2; B = 2pq$

Let a_2 be any non-zero irrational Gaussian number such that

$$a_0 * a_2 + 1 = s^2 \quad (1)$$

$$a_1 * a_2 + 1 = t^2 \quad (2)$$

Substituting

$$s = a_0 + r, t = a_1 + r \quad (3)$$

in (1) and (2),and subtracting one from the other, we get,

$$a_2 = (12A-20p+7)+i\sqrt{d}(12B-20q)$$

Thus, (a_0, a_1, a_2) is an irrational Gaussian Diophantine triple with property D(1).

If (a_0, a_1, a_2) is a Diophantine triple with property D(1) then the fourth tuple is given by

$$a_3 = (a_0 + a_1 + a_2) + 2[a_0 a_1 a_2 + rst] \tag{4}$$

where

$$a_0 a_1 + 1 = r^2, \quad a_0 a_2 + 1 = s^2, \quad a_1 a_2 + 1 = t^2$$

Employing the above formula (4) ,the fourth tuple is given by

$$\begin{aligned} a_3 &= \{432(\alpha^2 - d\beta^2) - 2160(p\gamma - dq\delta) + 4140\gamma - 3800\alpha + 1704A - 340p + 24\} \\ &+ i\sqrt{d}\{-862\alpha\beta - 2160[q\gamma + p\delta] + 4140\delta + 3800\beta + 1704B - 340q\} \end{aligned}$$

Where
$$\begin{aligned} &A = p^2 - dq^2, B = 2pq, \alpha = p^3 - 3pdq^2, \beta = dq^3 - 3p^2 q, \\ &\gamma = p^4 - 6dp^2q^2 + d^2q^4, \delta = 4p^3q - 4pdq^3 \end{aligned}$$

Observe that (a_0, a_1, a_2, a_3) is an irrational Gaussian Diophantine quadruple with property D(1).

SEQUENCE: II

Let
$$\begin{aligned} a_1 &= (3A - 8p + 5) + i\sqrt{d}(3B - 8q) \\ a_2 &= (12A - 20p + 7) + i\sqrt{d}(12B - 20q) \\ a_3 &= \{432(\alpha^2 - d\beta^2) - 2160(p\gamma - dq\delta) + 4140\gamma - 3800\alpha + 1704A - 340p + 24\} \\ &+ i\sqrt{d}\{-862\alpha\beta - 2160[q\gamma + p\delta] + 4140\delta + 3800\beta + 1704B - 340q\} \end{aligned}$$

Using the above formula (4) ,the fourth tuple is given by

$$\begin{aligned} a_4 &= \{62208(A\gamma^2 - Ad\delta^2 - 2dB\gamma\delta) - 580608(p\gamma^2 - pd\delta^2 - 2dq\gamma\delta) + 2360448(\gamma^2 - d\delta^2) \\ &- 5482368[(A^2 - dB^2)\alpha + 2dAB\alpha] + 8016720(\alpha^2 - d\beta^2) - 7665536(p\gamma - d\delta q) + 4817352\gamma \\ &- 1945632\alpha + 477291A - 63310p + 3432\} + i\sqrt{d}\{62208(B\gamma^2 - Bd\delta^2 - 2A\gamma\delta) \\ &- 580608(q\gamma^2 - qd\delta^2 + 2p\gamma\delta) + 4720896\gamma\delta - 10964736AB\alpha - 5482368(2Bd - A^2)\beta \\ &- 16033440\alpha\beta - 7665536(q\gamma + p\delta) + 4817352\delta + 1945632\beta + 477291B - 63310q\} \end{aligned}$$

We note that (a_1, a_2, a_3, a_4) is an irrational Gaussian Diophantine quadruple with property D(1).

Proceeding in this way, one may generate a sequence of irrational Gaussian Diophantine quadruples (a_0, a_1, a_2, a_3) , (a_1, a_2, a_3, a_4) ,……..with property D(1).

Some numerical examples are given below.

s.no	(p, q, d)	(a_0, a_1, a_2, a_3)	(a_1, a_2, a_3, a_4)

1	(1,1,2)	$\{(-5+i4\sqrt{2}),(-6-i2\sqrt{2}),(-25+i4\sqrt{2}),(3436+i4122\sqrt{2})\}$	$\{(-6-i2\sqrt{2}),(-25+i4\sqrt{2}),(3436+i4122\sqrt{2}),-(7795235+i47645723\sqrt{2})\}$
2	(1,1,3)	$\{(-8+i4\sqrt{3}),(-9-i2\sqrt{3}),(-37+i4\sqrt{3}),(5184+i4122\sqrt{3})\}$	$\{(-9-i2\sqrt{3}),(-37+i4\sqrt{3}),(5184+i4122\sqrt{3}),-(19916403+i58092059\sqrt{3})\}$
3	(1,1,5)	$\{(-14+i4\sqrt{5}),(-15-i2\sqrt{5}),(-61+i4\sqrt{5}),(8640+i4122\sqrt{5})\}$	$\{(-15-i2\sqrt{5}),(-61+i4\sqrt{5}),(8640+i4122\sqrt{5}),-(33194003+i78984731\sqrt{5})\}$

II: On Sequences of Irrational Gaussian Diophantine Quadruples with property D(4)

SEQUENCE: III

Let
$$\begin{aligned} a_0 &= A+3+i\sqrt{d}B \\ a_1 &= A-1+i\sqrt{d}B \end{aligned}$$

be such that

$$a_0 a_1 = [(A+1)+i\sqrt{d}(2q)]^2 = r^2 \quad \text{(say)}$$

Let a_2 be any non-zero irrational Gaussian number such that

$$a_0 * a_2 + 4 = s^2 \tag{5}$$

$$a_1 * a_2 + 4 = t^2 \tag{6}$$

Eliminating a_2 between (5) and (6), we get

$$a_1 s^2 - a_0 t^2 = 4(a_1 - a_0) \tag{7}$$

Using the linear transformations

$$\begin{aligned} s &= X + a_0 T \\ t &= X + a_1 T \end{aligned} \tag{8}$$

in (7), we get

$$X^2 = a_0 a_1 T^2 + 4$$

whose initial solution is

$T_0 = 1$ and $X_0 = A + 1 + i\sqrt{d}(2q)$

Using the above solutions in (8) and employing (5)

$$a_2 = (4A + 4) + i\sqrt{d}(8q)$$

Here we observe that (a_0, a_1, a_2) is an irrational Gaussian Diophantine triple with property D(4).

Employing Euler's solution, the fourth tuple is given by

$$a_3 = 4\{[A(A^2 + 3A + 2) - 12A(dp^2q^2)] - 12(dp^2q^2)] + i\sqrt{d}[A(3AB + 6B) + 2B - 8dp^3q^3]\}$$

Observe that (a_0, a_1, a_2, a_3) is an irrational Gaussian Diophantine quadruple with property D(4).

SEQUENCE: IV

Let

$$a_1 = (A - 1) + i\sqrt{d}B$$
$$a_2 = (4A + 4) + i\sqrt{d}(8q)$$
$$a_3 = 4\{[A(A^2 + 3A + 2) - 12A(dp^2q^2)] - 12(dp^2q^2)] + i\sqrt{d}[A(3AB + 6B) + 2B - 8dp^3q^3]\}$$

Proceeding as in sequence (III), the fourth tuples of this sequence are given by

$$a_4 = \{16A^5 + 48A^4 + 24A^3 - 24A^2 - 7A - 640A^3dp^2q^2 - 1152dp^2q^2A^2 + 1280d^2p^4q^4A - 288dp^2q^2A + 768d^2p^4q^4 + 96dp^2q^2 + 3\} + i\sqrt{d}\{160A^4pq + 384A^3pq + 144A^2pq - 96Apq - 1280A^2dp^3q^3 - 1536Adp^3q^3 + 512d^2p^5q^5 - 192dp^3q^3 + 336pq\}$$

We note that (a_1, a_2, a_3, a_4) is an irrational Gaussian Diophantine quadruple with property D(4).

Proceeding in this way, one may generate a sequence of irrational Gaussian Diophantine quadruples (a_0, a_1, a_2, a_3), (a_1, a_2, a_3, a_4) ,……..with property D(4)

Some numerical examples are given below:

s.no	(p,q,d)	(a_0,a_1,a_2,a_3)	(a_1,a_2,a_3,a_4)
1	(1,1,2)	$\{(2+i2\sqrt{2}),(-2+i2\sqrt{2})$ $,(8i\sqrt{2}),(-72i\sqrt{2})\}$	$\{(-2+i2\sqrt{2}),(i8\sqrt{2}),$ $(-i72\sqrt{2}),(-2310+i2528\sqrt{2})\}$
2	(1,1,3)	$\{(1+i2\sqrt{3}),(-3+i2\sqrt{3}),$ $(-4+i8\sqrt{3}),(144-i80\sqrt{3})\}$	$\{(-3+i2\sqrt{3}),(-4+i8\sqrt{3}),$ $(144-i80\sqrt{3}),(-12591-i1520\sqrt{3})\}$
3	(1,1,5)	$\{(-1+i2\sqrt{5}),(-5+i2\sqrt{5}),$ $(-12+i8\sqrt{5}),(624+i12\sqrt{5})\}$	$(-5+i2\sqrt{5}),(-12+i8\sqrt{5}),$ $(624+i12\sqrt{5})(4095-i40432\sqrt{5})$

3. CONCLUSION

In this paper, we have presented a method of constructing irrational Gaussian Diophantine quadruples being given an irrational Gaussian Diophantine pair such that their product added with either 1 or 4 is a perfect square.

To conclude, one may attempt to construct rational and irrational Gaussian Diophantine quadruples with suitable property and extend it to Diophantine quintuples and higher orders.

4. ACKNOWLEDGEMENT

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