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PASCAL TRIANGLE AND PYTHAGOREAN TRIPLES

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ABSTRACT

The concept of Pascal's triangle has fascinated mathematicians for several centuries. Similarly, the idea of Pythagorean triples prevailing for more than two millennia continue to surprise even today with its abundant properties and generalizations. In this paper, I have demonstrated ways through four theorems to determine Pythagorean triples using entries from Pascal's triangle.

Keywords: Binomial Coefficients. Pascal Triangle, Pythagorean Triple, Primitive Pythagorean Triple

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1. INTRODUTION

The concept of Pythagorean triples has been existing for more than two millennia known in the name of Greek philosopher and mathematician Pythagoras. The process of finding Pythagorean triples has been subject of great interest among amateur as well as trained mathematicians and has been subject of several research works. The concept of Pascal's triangle though became significant through French mathematician Blaise Pascal was known to ancient Indians and Chinese mathematicians as well. In this paper, I will determine methods of generating Pythagorean triples from the entries of Pascal's triangle in possible ways through theorems.

2. DEFINITIONS

2.1. Three positive integers a, b, c are said to constitute Pythagorean triple written in the form (a, b, c) if they satisfy the condition $c^2 = a^2 + b^2$ (2.1)

Geometrically, the triple constituting the numbers (a, b, c) satisfying (2.1) forms three sides of a right triangle in which c is its hypotenuse and a, b being its other two sides (legs).

2.2. The numbers of the form $\binom{n}{r} = \frac{n!}{r! \times (n-r)!}$ (2.2) where $0 \le r \le n$ are said to be Binomial coefficients since they form coefficients of the binomial expansion $(x+y)^n$. These numbers exist as entries of Pascal's triangle.

I now present ways to generate Pythagorean triples through binomial coefficients which are entries of Pascal's triangle through the following theorems.

3. THEOREM 1

Let $\alpha = \binom{n}{0}, \beta = \binom{n}{1}$. Then for $n \geq 1$, the Pythagorean triple (*a*, *b*, *c*) is given by

$$a = \alpha + 2\beta,\ b = 2\beta(\alpha + \beta), c = b + 1 \ (3.1)$$

Proof: Using the definition of the binomial coefficient from (2.2), we have

$$a = \alpha + 2\beta = \binom{n}{0} + 2\binom{n}{1} = 1 + 2n, b = 2\beta(\alpha + \beta) = 2\binom{n}{1}\left(\binom{n}{0} + \binom{n}{1}\right) = 2n(1+n)$$

$$a^2 + b^2 = (2n+1)^2 + 4n^2(n+1)^2 = 4n^4 + 8n^3 + 8n^2 + 4n + 1 = \left(2n^2 + 2n + 1\right)^2$$

$$= \left[2n(n+1)+1\right]^2 = (b+1)^2 = c^2$$

Thus, (*a*, *b*, *c*) forms Pythagorean triple and this completes the proof.

4. THEOREM 2

For $n \geq 2$, let $\alpha = \binom{n}{1}, \beta = \binom{n}{2}$, Then the Pythagorean triple (*a*, *b*, *c*) is given by

$$a = 2\alpha - 1,\ b = 4\beta,\ c = b + 1 \ \ (4.1)$$

Proof: First, we notice that $\alpha = \binom{n}{1} = n, \beta = \binom{n}{2} = \frac{n(n-1)}{2}$

$$a^2 + b^2 = (2n-1)^2 + 4n^2(n-1)^2 = 4n^4 - 8n^3 + 8n^2 - 4n + 1 = \left(2n^2 - 2n + 1\right)^2$$

$$= \left[2n(n-1)+1\right]^2 = (b+1)^2 = c^2$$

Thus, (*a*, *b*, *c*) forms Pythagorean triple and this completes the proof.

In the following theorem, I will present a more general result to generate Pythagorean triples from particular entries of Pascal's triangle.

5. THEOREM 3

Let $\alpha = \binom{n}{r}$, and let $r, k \geq 2$ be two natural numbers. Then the Pythagorean triple (*a*, *b*, *c*) is given by

$$a = 2r^k\alpha,\ b = r\left(r^{2k-2} - 1\right)\alpha,\ c = r\left(r^{2k-2} + 1\right)\alpha \ \ (5.1)$$

Proof: From the given values of *a*, *b* we see that

$$a^2+b^2=4r^{2k}\alpha^2+r^2\left(r^{2k-2}-1\right)^2\alpha^2=\alpha^2\left[4r^{2k}+r^{4k-2}-2r^{2k}+r^2\right]$$
$$=\alpha^2r^2\left[r^{4k-4}+2r^{2k-2}+1\right]=\left[\alpha r\left(r^{2k-2}+1\right)\right]^2=c^2$$

Since *a*, *b* and *c* are integers, from (2.1), it follows that (*a*, *b*, *c*) is a Pythagorean triple. This completes the proof.

6. THEOREM 4

For any two integers $r,k\geq 2$, the primitive Pythagorean triples are given by

$a=r^{k-1},b=\dfrac{r^{2k-2}-1}{2},c=\dfrac{r^{2k-2}+1}{2}$ (6.1) if *r* is odd and

$a=2r^{k-1},b=r^{2k-2}-1,c=r^{2k-2}+1$ (6.2) if *r* is even.

Proof: With respect to either *r* is even or odd, according to given expressions, it is straightforward to verify that $a^2+b^2=c^2$. Moreover, from (6.2), if *r* is even, then it is clear that *a*, *b*, *c* are integers. If *r* is odd, then $a=r^{k-1}$ is clearly an odd integer. Now the numerator of *b* and *c* namely $r^{2k-2}-1$ and $r^{2k-2}+1$ becomes even since if *r* is odd, r^{2k-2} is odd. Hence *b* and *c* according to the expressions given by (6.1) must be integers. Moreover, the greatest common divisor of *a*, *b*, *c* is 1. Hence in either case, (*a*, *b*, *c*) forms a primitive Pythagorean triple. This completes the proof.

7. ILLUSTRATIONS

In this section, I present some illustrations to generate Pythagorean triples as discussed in the three theorems and corollary in sections 3 to 6.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
. . . .

Figure 1 Pascal's Triangle with first two entries highlighted beginning with first row

As in theorem 1, if we consider the first two entries of Pascal's triangle beginning from row 1, as shown in Figure 1, we can generate the following Pythagorean triples using (3.1)

For n = 1, we have $\alpha = 1, \beta = 1, a = \alpha + 2\beta = 3, b = 2\beta(\alpha + \beta) = 4, c = b + 1 = 5$

For n = 2, we have $\alpha = 1, \beta = 2, a = \alpha + 2\beta = 5, b = 2\beta(\alpha + \beta) = 12, c = b + 1 = 13$

For n = 3, we have $\alpha = 1, \beta = 3, a = \alpha + 2\beta = 7, b = 2\beta(\alpha + \beta) = 24, c = b + 1 = 25$

For n = 4, we have $\alpha = 1, \beta = 4, a = \alpha + 2\beta = 9, b = 2\beta(\alpha + \beta) = 40, c = b + 1 = 41$

For n = 5, we have $\alpha = 1, \beta = 5, a = \alpha + 2\beta = 11, b = 2\beta(\alpha + \beta) = 60, c = b + 1 = 61$

For n = 6, we have $\alpha = 1, \beta = 6, a = \alpha + 2\beta = 13, b = 2\beta(\alpha + \beta) = 84, c = b + 1 = 85$

For n = 7, we have $\alpha = 1, \beta = 7, a = \alpha + 2\beta = 15, b = 2\beta(\alpha + \beta) = 112, c = b + 1 = 113$

1
1 1
1 **2** **1**
1 **3** **3** 1
1 **4** **6** 4 1
1 **5** **10** 10 5 1
1 **6** **15** 20 15 6 1
1 **7** **21** 35 35 21 7 1
1 **8** **28** 56 70 56 28 8 1
- - - - - - - - - - - - - - - -

Figure 2 Pascal's Triangle with second and third entries highlighted beginning with second row

As in theorem 2, if we consider second and third entries of Pascal's triangle beginning from row 2, as shown in Figure 2, we can generate the following Pythagorean triples using (4.1)

For n = 2, we have $\alpha = 2, \beta = 1, a = 2\alpha - 1 = 3, b = 4\beta = 4, c = b + 1 = 5$

For n = 3, we have $\alpha = 3, \beta = 3, a = 2\alpha - 1 = 5, b = 4\beta = 12, c = b + 1 = 13$

For n = 4, we have $\alpha = 4, \beta = 6, a = 2\alpha - 1 = 7, b = 4\beta = 24, c = b + 1 = 25$

For n = 5, we have $\alpha = 5, \beta = 10, a = 2\alpha - 1 = 9, b = 4\beta = 40, c = b + 1 = 41$

For n = 6, we have $\alpha = 6, \beta = 15, a = 2\alpha - 1 = 11, b = 4\beta = 60, c = b + 1 = 61$

For n = 7, we have $\alpha = 7, \beta = 21, a = 2\alpha - 1 = 13, b = 4\beta = 84, c = b + 1 = 85$

For n = 8, we have $\alpha = 8, \beta = 28, a = 2\alpha - 1 = 15, b = 4\beta = 112, c = b + 1 = 113$

As illustration of equation (5.1) of theorem 3, if we choose the entry 35 located in seventh row (see Figure 2), we have $\alpha = 35 = \binom{7}{3} = \binom{7}{4}$. Here $r = 3$ or 4.

For $\alpha = 35, r = 3$ using (5.1), we can generate the following Pythagorean triples of the form $a = 2r^k\alpha = 70 \times 3^k, b = r\left(r^{2k-2}-1\right)\alpha = 105\left(3^{2k-2}-1\right), c = r\left(r^{2k-2}+1\right)\alpha = 105\left(3^{2k-2}+1\right)$.

Since $r = 3$ is odd, by (6.1), the primitive Pythagorean triples are given by

$$a = 3^{k-1}, b = \frac{3^{2k-2}-1}{2}, c = \frac{3^{2k-2}+1}{2}$$

For $k = 2$, we have $a = 3, b = \frac{3^2-1}{2} = 4, c = \frac{3^2+1}{2} = 5$

For $k = 3$, we have $a = 3^2 = 9, b = \frac{3^4-1}{2} = 40, c = \frac{3^4+1}{2} = 41$

For $k = 4$, we have $a = 3^3 = 27, b = \frac{3^6-1}{2} = 364, c = \frac{3^6+1}{2} = 365$

Similarly, for $\alpha = 35, r = 4$ using (5.1), we can generate the following Pythagorean triples of the form $a = 2r^k\alpha = 70 \times 4^k, b = r\left(r^{2k-2}-1\right)\alpha = 140\left(4^{2k-2}-1\right), c = r\left(r^{2k-2}+1\right)\alpha = 140\left(4^{2k-2}+1\right)$.

Since $r = 4$ is even, by (6.2), the primitive Pythagorean triples are given by

$$a = 2 \times 4^{k-1}, b = 4^{2k-2}-1, c = 4^{2k-2}+1$$

For $k = 2$, we have $a = 8, b = 4^2 - 1 = 15, c = 4^2 + 1 = 17$

For $k = 3$, we have $a = 32, b = 4^4 - 1 = 255, c = 4^4 + 1 = 257$

For $k = 4$, we have $a = 128, b = 4^6 - 1 = 4095, c = 4^6 + 1 = 4097$

Similarly, by choosing any particular entry from Pascal triangle and finding corresponding value of r, we can determine several Pythagorean triples.

8. CONCLUSION

The main purpose of this paper is to connect the concepts of generating Pythagorean triples with the entries of Pascal's triangle.

In theorem 1, by choosing the first two entries of the Pascal's triangle and beginning from row 1 as shown in Figure 1, I had generated Pythagorean triples given by (3.1). Similarly, in theorem 2, beginning with second row, choosing second and third entries as shown in Figure 2, I had generated Pythagorean triples using the formulas given by (4.1).

In section 5, in theorem 3, by choosing any entry of Pascal's triangle located at third place or higher place in a particular row, I had provided formulas for generating Pythagorean triples through equation (5.1). It is to be noted that the triples generated through (5.1) are not primitive. To generate primitive Pythagorean triple, I had provided theorem 4, in section 6, in which depending on the parity of r, two formulas are provided through equations (6.1) and (6.2). In particular, if r is odd, then the primitive Pythagorean triple is given by (6.1) and if r is even, then they are given by (6.2). Thus by choosing different values of $r, k \geq 2$ we can generate as many primitive Pythagorean triples as possible. In section 7, several illustrations were provided to explain the formulas obtained in theorems 1 to 4 of this paper. These calculations provided various Pythagorean triples as expected through the theorems established.

Thus, by proving four new theorems in this paper, I had exhibited the connection between the entries of Pascal's triangle and generation of Pythagorean triples, the two fascinating and everlasting concepts in mathematics.

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