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# ON THE EXISTENCE OF UNIQUE COMMON FIXED POINT OF TWO MAPPINGS IN A METRIC-LIKE SPACE

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## ABSTRACT

The main objective of this article is to introduce a fixed-point result involving two mappings satisfying contraction in a metric-like space. The article has been designed in the following manner. In the first section the authors have mentioned some definitions and commonly used notations. In the second section, they have mentioned some fixed-point results. In the third section the authors have introduced their main result and in the last section, using these results, the authors have obtained some conditions that assure the existence of a common fixed point of a pair of mappings.

**Keywords:** Metric-Like Space, 0-Complete Metric-Like Space, Common Fixed Point, F-Contraction

## 1. INTRODUCTION

In 1922, Banach (1922) introced his result, known as Banach's Fixed Point Theorem. After that several mathematicians worked on this result and made some successful attemps to generalize his idea in other ways. Recently in 2012, as a generalization of Banach's Contraction Condition, Wardowski (2012) introduced a new type of contraction condition, called an $F$-contraction. After this introduction, several mathemticians have widely used this idea to introduce some interesting results of fixed point. Beside these generalizations, some other mathematicians tried to generalize the notion of a Metric Space. Partial Metric Space Matthews (1994), Metric-Like Space Amini-Harandi (2012)are some notable generalizatons of the idea of a Metric Space. During these years several results on fixed point have been introduced on these spaces.

In our paper we'll consider the Metric-Like Space and the $F$-contraction to investigate the existence of a unique common fixed point for a pair of mapping. Foe that we'll mention some definitions and results firts.

The idea of a Metric Space was introduced by M. Fréchet in the year 1906. Through this idea of Metric Space, he tried to define the distance between two points of an arbitrary set in an abstract manner as follows,

**Definition 1.1** *(Metric Space) Suppose* $X \neq \emptyset$ *and* $d: X^2 \to [0, \infty)$ *be a mapping such that*

1) $d(x, y) \geq 0; \quad \forall x, y \in X.$
2) $d(x, y) = 0 \Leftrightarrow \quad x = y.$
3) $d(x, y) = d(y, x); \quad \forall x, y \in X.$
4) $d(x, y) \leq d(x, z) + d(z, y); \quad \forall x, y, z \in X.$

then the mapping $d$ will be called a Metric or a Distance Function on $X$ and the ordered pair $(X, d)$ will be called a Metric Space.

**Example 1.2** *The set* $R$ *of all real numbers forms metric space with respect to metric* $d_u$*, defined by* $d_u(x, y) = |x - y|, \forall x, y \in R.$

**Example 1.3** *The set* $C$ *of all complex numbers forms metric space with respect to metric* $d_u$*,defined by* $d_u(z, w) = |z - w| = \sqrt{(a-c)^2 + (b-d)^2}$ *, where* $z = a + ib, w = c + id \in C$

In 1994, S.G. Matthew introduced a generalization of metric space and referred it as a Partial Metric Space.

**Definition 1.4** *(Partial Metric Space)* Matthews (1994) *Suppose* $X \neq \emptyset$ *and* $p: X^2 \to [0, \infty)$ *be a mapping such that*

1) $p(x, x) = p(x, y) = p(y, y) \Rightarrow x = y; \forall x, y \in X.$
2) $p(x, x) \leq p(x, y); \forall x, y \in X.$
3) $p(x, y) = p(y, x); \forall x, y \in X.$
4) $p(x, y) \leq p(x, z) + p(y, z) - p(z, z); \forall x, y, z \in X.$

then the mapping $p$ will be called a partial metric on $X$ and the ordered pair $(X, p)$ will be called a Partial Metric Space.

**Example 1.5** *In the set* $R$ *of real numbers forms a partial metric space with respect to the mapping* $p(x, y) = |x - y|; \forall x, y \in R.$

**Example 1.6** *In the set* $R$ *of real numbers forms a partial metric space with respect to the mapping* $p(x, y) = max\{|x|, |y|\}; \forall x, y \in R.$

In 2012, A.A. Harandi introduced a new generalization of metric space called a Metric-Like space

**Definition 1.7** *(Metric-Like Space)* Amini-Harandi (2012) *Suppose $X \neq \emptyset$ and $d': X^2 \to [0, \infty)$ be a mapping such that*

1) $d'(x, y) \geq 0; \quad \forall x, y \in X.$
2) $d'(x, y) = 0 \Rightarrow \quad x = y.$
3) $d'(x, y) = d(y, x); \quad \forall x, y \in X.$
4) $d'(x, y) \leq d(x, z) + d(z, y); \quad \forall x, y, z \in X.$

then the ordered pair $(X, d)$ will be called a Metric-Like Space or a Dislocated Metric Space.

**Example 1.8** *In the set $R$ of real numbers forms a metric-like space with respect to the mapping $d'(x, y) = max\{|x|, |y|\}; \forall x, y \in R.$*

**Example 1.9** *In the set $R$ of real numbers forms a metric-like space with respect to the mapping $d'(x, y) = |x| + |y|; \forall x, y \in R.$*

**Example 1.10** *The set $C[a, b]$ of all real-valued continuous functions defined on a compact interval $[a, b]$ forms a metric-like space with respect to te mapping $d'(f, g) = sup_{t \in [a,b]}(|f(t)| + |g(t)|); \forall f, g \in C[a, b].$*

**Remark 1.11** *Clearly every Metric Space is a Partial Metric Space as well as a Metric-Like Space. But 1.6, 1.8 shows that the converse is not true.*

**Remark 1.12** *Every Partial Metric Space is a Metric-Like Space but 1.9 shows that the converse is not true.*

**Definition 1.13** *(Convergence of a Sequence)* Amini-Harandi (2012) *In a metric-like space $(X, d')$ suppose $\{x_n\}$ be a sequence. Then $\{x_n\}$ is said to converge to some limit $x$ if $lim_{n \to \infty} d'(x_n, x) = d'(x, x)$*

**Definition 1.14** *(Cauchy Sequence and Completeness)* Amini-Harandi (2012)

1. In a metric-like space $(X, d')$ a sequence $\{x_n\}$ is said to be a Cauchy Sequence if $\lim_{m,n \to \infty} d'(x_m, x_n)$ exists finitely.
2. A metric-like space $(X, d')$ is said to be a complete metric like space if for every Cauchy Sequence $\{x_n\}$ in $X$ there exists $x \in X$ such that $\lim_{m,n \to \infty} d'(x_m, x_n) = \lim_{n \to \infty} d'(x_n, x) = d'(x, x)$.

**Remark 1.15** *In a metric-like space the limit of a sequence may not be unique. For example in the space $(R, d')$ where $d'(x, y) = ma\{|x|, |y|\}$ consider the sequence $x_n = \frac{1}{n}$. Then for any real number $x \geq 1$, $d'(x_n, x) = max\{\frac{1}{n}, x\} = x = max\{x, x\}$.*

**Remark 1.16** *In a metric-like space a convergent sequence may not be a Cauchy sequence.*

**Remark 1.17** *In a complete metric like space if a sequence $\{x_n\}$ is a Cauchy sequence such that $lim_{m,n\to\infty} d'(x_m, x_n) = 0$ then its limit will be unique. For this, suppose on the contrary that the limit of the sequence is not unique. Then there will exist $x, x'$ with the following properties*

$$\lim_{n\to\infty} d'(x_n, x) = d'(x, x) = \lim_{m,n\to\infty} d'(x_m, x_n) = 0,$$
$$\lim_{n\to\infty} d'(x_n, x') = d'(x', x') = \lim_{m,n\to\infty} d'(x_m, x_n) = 0,$$
$$d'(x, x') \neq 0.$$

Then $0 < d'(x, x') \leq d'(x_n, x) + d'(x_n, x') \to 0$ — a contradiction

The above fact leads to the following definition,

**Definition 1.18** *(0-Cauchy Sequence and 0-Complete Space)* Shukla et al. (2013)

- In a metric like space $(X, d')$ a sequence $\{x_n\}$ will be called a 0-Cauchy sequence if $\lim_{m,n\to\infty} d'(x_m, x_n) = 0$.
- A metricclike space $(X, d')$ is said to be a 0-Complete metric like space if every 0-Cauchy sequence in $X$ converges to some point $x \in X$ such that $d'(x, x) = 0$.

Clearly every 0-Cauchy sequence ia a Cauchy sequence and a complete metric like space is a 0-complete metric like space.

**Definition 1.19** *(Coincidence Point and Point of Coincidence)* Jungck (1996) *Suppose $X \neq \emptyset$ and $f, g: X \to X$ be two functions. A point $w \in X$ is said to be a point of coincidence of $f$ and $g$ if there exists $x \in X$ such that*

$$fx = gx = w$$

The point $x \in X$ is called a *coincidence* point of $f$ and $g$.

**Definition 1.20** *(Weakly Compatible Mapping)* Jungck (1996) *Suppose $X \neq \emptyset$ and $f, g: X \to X$ be two functions. Then $f$ and $g$ are said to be weakly compatible if they commutes at their coincidence points i.e. if $f(x) = g(x)$, for some $x \in X$ then $fg(x) = gf(x)$.*

In 2012, D. Wardowski introduced the idea of $F$-contraction in the following manner;

**Definition 1.21** *($F$-Contraction)* Wardowski (2012) *Suppose, $F: (0, \infty) \to (-\infty, \infty)$ be a mapping such that*

1. $F$ is strictly increasing.
2. $\lim_{x\to 0^+} F(x) = -\infty$.
3. $\exists m \in (0,1)$ such that $\lim_{x\to 0^+} x^m F(x) = 0$.

The followings are some examples of such functions

**Example 1.22** $F(t) = \log t$

**Example 1.23** $F(t) = t\log t$

**Example 1.24** $F(t) = -\frac{1}{\sqrt{t}}$.

- Now suppose $(X,d')$ be a metric-like space and $T: X \to X$ be a mapping such that whenever $d'(Tx,Ty) > 0$, $\exists \tau > 0$ such that

$$\tau + F(d'(Tx,Ty)) \leq F(d'(x,y))$$

then $T$ is said to be an $F$-contraction defined on $X$.

**Notations:**

In our paper we'll commonly use the folloing notations

1) $\mathcal{F} = \{F: (0,\infty) \to (-\infty,\infty): F$ satisfies the three conditons mentioned in Definition 1.21$\}$
2) For a mapping $f: (X,d') \to (X,d')$,

$$M(x,y) = max\{d'(x,y), d'(x,Tx), d'(y,Ty), d'(x,Ty), d'(y,Tx), d'(x,x), d'(y,y)\}$$

3) For mappings $f, g: (X,d') \to (X,d')$,

$$N_2(x,y) = max\{d'(gx,gy), d'(fx,gx), d'(fy,gy), d'(gx,fy), d'(fx,gy), d'(gx,gx), d'(gy,gy)\}$$

4) For mappings $f, g: (X,d') \to (X,d')$,

$$M_2(x,y) = max\left\{d'(x,y), d'(x,fx), d'(y,gy), \frac{d'(fx,y) + d'(x,gy)}{4}\right\}$$

## 2. SOME RESULTS

**Lemma 2.1** Karapinar and Salimi (2013) *In a metric-like space $(X,d')$ the following results hold*

1) $x \neq y \Rightarrow d'(x,y) > 0$.
2) $d'(x,y) = 0 \Rightarrow d'(x,x) = d'(y,y) = 0$.
3) If a sequence $\{x_n\}$ converges to $x \in X$ such that $d'(x,x) = 0$, then for all $y \in X$ $d'(x_n,y)$ converges to $\{d'(x_n,y)\}$.
4) If $\{x_n\}$ be a sequence in $X$ such that $d'(x_n,x_{n+1}) \to 0$ as $n \to \infty$, then $d'(x_n,x_n), d'(x_{n+1},x_{n+1}) \to 0$ as $n \to \infty$.
5) Suppose $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} d'(x_n,x_{n+1}) = 0$. If $\lim_{m,n\to\infty} d'(x_m,x_n) \neq 0$ then there exists $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that the following sequences

$$\{d'(x_{n_k},x_{m_k}\}, \{d'(x_{n_{k-1}},x_{m_k})\}, \{d'(x_{n_k},x_{m_k-1})\}, \{d'(x_{n_k+1},x_{m_k})\}, \{d'(x_{n_k},x_{m_k+1})\}$$

will converge to $\epsilon$ as $k \to \infty$

**Theorem 2.2** Amini-Harandi (2012) *Suppose $(X,d')$ be a complete metric-like space and $T: X \to X$ be a map such that $\forall x,y \in X$*

$$d'(Tx,Ty) \leq \psi(M(x,y))$$

where $\psi: [0,\infty) \to [0,\infty)$ is a non-decreasing function such that

1) $\psi(t) < t; \forall t > 0$.
2) $\lim_{s\to t^+} \psi(s) < t; \forall t > 0$.
3) $\lim_{t\to\infty}(t - \psi(t)) = \infty$.

Then $T$ has a fixed point.

**Theorem 2.3** Amini-Harandi (2012) *Suppose $(X,d')$ be a complete metric-like space and $T: X \to X$ be a map such that $\forall x,y \in X$*

$$d'(Tx, Ty) \leq d'(x, y)\phi(d'(x, y))$$

where $\phi: [0, \infty) \to [0, \infty)$ is a non-decreasing continuous function such that $\phi(t) = 0 \Leftrightarrow t = 0$. Then $T$ has a unique fixed point.

**Theorem 2.4** Karapinar and Salimi (2013) *Suppose* $(X, d')$ *be a complete metric-like space and* $f, g: X \to X$ *be a map such that* $\forall x, y \in X$

$$d'(fx, fy) \leq \psi(N_2(x, y))$$

where $\psi: [0, \infty) \to [0, \infty)$ is a non-decreasing function such that

1) $\psi(t) < t; \forall t > 0$.
2) $\lim_{s \to t^+} \psi(s) < t; \forall t > 0$.
3) $\lim_{t \to \infty}(t - \psi(t)) = \infty$.

If the range of $g$ contains the range of $f$ and $f(X)$ or $g(X)$ is a closed subset of X, then $f$ and $g$ will have a unique point of coincidence in X. Moreover if the mappings are weakly compatible, then they will have a unique common fixed point $z \in X$ such that $d'(z, z) = 0$.

For more fixed point results in a metric-like space we refer Amini-Harandi (2012), Karapinar and Salimi (2013), Shukla et al. (2013), Fabijano et al.(2020).

## 3. MAIN RESULT

We now introduce our main result;

**Theorem 3.1** *Suppose* $(X, d')$ *be a 0-complete metric like space and* $f, g: X \to X$ *be two functions such that whenever* $d'(fx, gy) > 0$ *there exists* $\tau > 0$ *such that*

$$\tau + F(d'(fx, gy)) \leq F(M_2(x, y)) \qquad \text{Equation 1}$$

for some $F \in \mathcal{F}$. Then $f$ and $g$ will have a unique common fixed point in X.

*Proof.* Suppose $x_0 \in X$ and define a sequence $\{x_n\}$ as follows

$$\begin{aligned} fx_0 &= x_1; \\ gx_1 &= x_2; \\ fx_2 &= x_3; \\ gx_4 &= x_5; \\ &\vdots \end{aligned}$$

i.e. in general

$$fx_{2n} = x_{2n+1}; \qquad gx_{2n+1} = x_{2n+2}; \forall n \geq 0.$$

Further let us denote by

$$d'_n = d'(x_n, x_{n+1}).$$

Now let us consider the following cases

**Case 1:** Suppose $d'_n = 0$ for some $n$, then $d'(x_n, x_{n+1}) = 0$.
If $n = 2m$ then,

$$d'(x_{2m}, x_{2m+1}) = 0$$
$$\Rightarrow \quad x_{2m} = x_{2m+1}$$
$$\Rightarrow \quad gx_{2m-1} = fx_{2m} = fgx_{2m-1}$$

Therefore $x_{2m} = gx_{2m-1}$ is a fixed point of $f$.
Now if $d'(gx_{2m}, x_{2m}) > 0$, then we have Equation 1 we have

$$\tau + F(d'(fx_{2m}, gx_{2m})) \le F(M_2(x_{2m}, x_{2m})) \qquad \text{Equation 2}$$

Where,

$$M_2(x_{2m}, x_{2m}) = \max\left\{d'(x_{2m}, x_{2m}), d'(x_{2m}, fx_{2m}), d'(x_{2m}, gx_{2m}), \frac{d'(x_{2m}, gx_{2m}) + d'(fx_{2m}, x_{2m})}{4}\right\}$$
$$= \max\left\{d'(x_{2m}, x_{2m+1}), d'(x_{2m}, x_{2m+1}), d'(x_{2m}, gx_{2m}), \frac{d'(x_{2m}, gx_{2m}) + d'(x_{2m+1}, x_{2m})}{4}\right\}$$
$$= d'(x_{2m}, gx_{2m})$$

Therefore from Equation 2 we have

$$\tau + F(d'(x_{2m}, gx_{2m})) = \tau + F(d'(fx_{2m}, gx_{2m})) \le F(d'(x_{2m}, gx_{2m}))$$

— a contradiction.

Hence $d'(gx_{2m}, x_{2m}) = 0$ i.e. $x_{2m}$ is the fixed point of $g$ and consequently the common fixed point of $f$ and $g$. If $n = 2m + 1$, then proceeding in the similar way we can prove that $f$ and $g$ will have a common fixed point.

Therefore if $d'_n = 0$ for some $n$ then $f$ and $g$ will have a common fixed point.

**Case 2:** Now let us assume that $d'_n \neq 0, \forall n \geq 0$.
Then for $x = x_{2n}$ and $y = x_{2n+1}$ we have from Equation 1

$$\tau + F(d'(fx_{2n}, gx_{2n+1})) \le F(M_2(x_{2n}, x_{2n+1})) \qquad \text{Equation 3}$$

Where,

$$M_2(x_{2n}, x_{2n+1})$$
$$= \max\left\{d'(x_{2n}, x_{2n+1}), d'(x_{2n}, x_{2n+1}), d'(x_{2n+1}, x_{2n+2}), \frac{d'(x_{2n}, x_{2n+2}) + d'(x_{2n+1}, x_{2n+1})}{4}\right\}$$
$$\le \max\left\{\begin{array}{c} d'(x_{2n}, x_{2n+1}), d'(x_{2n}, x_{2n+1}), d'(x_{2n+1}, x_{2n+2}), \\ \frac{d'(x_{2n}, x_{2n+1}) + d'(x_{2n+1}, x_{2n+2}) + d'(x_{2n}, x_{2n+1}) + d'(x_{2n}, x_{2n+1})}{4} \end{array}\right\}$$
$$= \max\left\{d'(x_{2n}, x_{2n+1}), d'(x_{2n}, x_{2n+1}), d'(x_{2n+1}, x_{2n+2}), \frac{3d'(x_{2n}, x_{2n+1}) + d'(x_{2n+1}, x_{2n+2})}{4}\right\}$$
$$= \max\{d'(x_{2n}, x_{2n+1}), d'(x_{2n+1}, x_{2n+2})\}$$

If $M_2(x_{2n}, x_{2n+1}) = d'(x_{2n+1}, x_{2n+2})$, then Equation 3 will imply

$$\tau + F(d'(fx_{2n}, gx_{2n+1})) = \tau + F(d'(x_{2n+1}, x_{2n+2})) \leq F(d'(x_{2n+1}, x_{2n+2}))$$

Equation 4

—a contradiction.

Thus

$$M_2(x_{2n}, x_{2n+1}) = d'(x_{2n}, x_{2n+1})$$
$$\Rightarrow \quad d'(x_{2n+1}, x_{2n+2}) < d'(x_{2n}, x_{2n+1})$$
$$\Rightarrow \quad d'_{2n+1} < d'_{2n}$$

Similarly, condsidering $x = x_{2n+2}$ and $y = x_{2n+1}$ in Equation 1 we can prove that

$$d'_{2n+2} < d'_{2n+1}$$

Therefore

$$d'_{2n+2} < d'_{2n+1} < d'_{2n}$$

Thus $\{d'_n\}$ is a monotonic decreasing sequence. Since it is bounded below by 0, it is convergent. Suppose the

$$\lim_{n\to\infty} d'_n = l \geq 0$$

.

Now if $l > 0$, then taking limit as $n \to \infty$ on both sides of (??) we have

$$\tau + F(l) \leq F(l)$$

— a contracdiction. Therefore

$$l = \lim_{n\to\infty} d'_n = \lim_{n\to\infty} d'(x_n, x_{n+1}) = 0$$

Now we claim that $\{x_n\}$ is a 0-Cauchy Sequence i.e. $\lim_{m,n\to\infty} \mathrm{d}'(x_\mathrm{n}, x_m) = 0$. For this on the contrary, let us assume that $\{x_n\}$ is not a 0-Cauchy Sequence. Then from *Lemma 2.1* we have, $\exists \epsilon > 0$ and two subsequences $\{x_{n_k}\}, \{x_{m_k}\}$ of $\{x_n\}$ such that the following sequences

$$\{d'(x_{n_k}, x_{m_k})\}, \quad \{d'(x_{n_k-1}, x_{m_k})\}, \quad \{d'(x_{n_k}, x_{m_k-1})\}, \{d'(x_{n_k+1}, x_{m_k})\}, \quad \{d'(x_{n_k}, x_{m_k+1})\}$$

converges to $\epsilon$.

Now if $n_k$ and $m_k$ are both even then taking $x = x_{n_k}$ and $y = x_{m_k+1}$ we have

$$\tau + F(d'(fx_{n_k}, gx_{m_k+1})) \leq F(M_2(x_{n_k}, x_{m_k+1}))$$

Where,

$$M_2(x_{n_k}, x_{m_k+1})$$

$$= \max\left\{d'\left(x_{n_k}, x_{m_k+1}\right), d'\left(x_{n_k}, fx_{n_k}\right), d'\left(x_{m_k+1}, gx_{m_k+1}\right), \frac{d'\left(x_{n_k}, gx_{m_k+1}\right)+d'\left(fx_{n_k}, x_{m_k+1}\right)}{4}\right\}$$

$$= \max\left\{d'\left(x_{n_k}, x_{m_k+1}\right), d'\left(x_{n_k}, x_{n_k+1}\right), d'\left(x_{m_k+1}, x_{m_k+2}\right), \frac{d'\left(x_{n_k}, x_{m_k+2}\right) + d'\left(x_{n_k+1}, x_{m_k+1}\right)}{4}\right\}$$

$$\leq \max\left\{\begin{array}{c} d'\left(x_{n_k}, x_{m_k+1}\right), d'\left(x_{n_k}, x_{n_k+1}\right), d'\left(x_{m_k+1}, x_{m_k+2}\right), \\ \frac{d'\left(x_{n_k}, x_{m_k+1}\right) + d'\left(x_{m_k+1}, x_{m_k+2}\right) + d'\left(x_{n_k+1}, x_{n_k}\right) + d'\left(x_{n_k}, x_{m_k+1}\right)}{4} \end{array}\right\}$$

$$\to \quad \epsilon \quad as \quad k \to \infty$$

Taking $k \to \infty$ in the above inequality we have

$$\tau + F(\epsilon) < F(\epsilon)$$

— a contradiction as $\epsilon > 0$.

If $n_k$ and $m_k$ are both odd or one is even and other is odd then choosing suitable terms as $x$ and $y$ as above we will arrive at a contradiction.

This proves that $\{x_n\}$ is a 0-Cauchy Sequence. Since $X$ is a 0-complete metric like space, thus $\{x_n\}$ is converges to $x' \in X$ with $d'(x', x') = 0$.

Now if $d'(fx', x') > 0$ then $fx' \neq x'$.

Since $\{x_n\}$ is a 0-Cauchy sequence, thus uniqueness of limit implies that $\{x_n\}$ does not converge to $fx'$. Taking $x = x'$ and $y = x_{2n+1}$ in Equation 1 we have

$$\tau + F(d'(fx', gx_{2n+1})) \leq F(M_2(x', x_{2n+1}))$$

Where,

$$M_2(x', x_{2n+1})$$

$$= \max\left\{d'(x', x_{2n+1}), d'(x', fx'), d'(x_{2n+1}, gx_{2n+1}), \frac{d'(x', gx_{2n+1}) + d'(x_{2n+1}, fx')}{4}\right\}$$

$$\to \quad d'(x', fx') \quad as \quad n \to \infty$$

Thus taking $n \to \infty$ in the above inequality we have

$$\tau + F(d'(fx', x)) \leq F(d'(x', fx'))$$

— a contradiction.

Therefore

$$d'(fx', x') = 0 \Rightarrow fx' = x'.$$

Thus $x'$ is a fixed point of $f$.

Now if $d'(fx', gx') > 0$ then taking $x = y = x'$ in Equation 1 we have,

$$\tau + F(d'(fx', gx')) \leq F(M_2(x', x'))$$

where

$$M_2(x', x') = \max\left\{d'(x', x'), d'(x', fx'), d'(x', gx'), \frac{d'(x', gx') + d'(x', fx')}{4}\right\}$$
$$= d'(x', gx')$$

Therefore from the above inequality we have

$$\tau + F(d'(x', gx')) = \tau + F(d'(fx', gx')) \leq F(d'(x', gx'))$$

— a contradiction. Therefore

$$d'(gx', x') = 0 \Rightarrow gx' = x'.$$

Thus $x'$ is a fixed point of $g$.

Therefore $f$ and $g$ has a common fixed point.

**Uniqueness:** To prove the uniqueness let us assume on the contrary that there exists two common fixed points $x'$ and $x''$ of $f$ and $g$. Then $d'(x', x'') > 0$. Then taking $x = x'$ and $y = x''$ in Equation 1 we have

$$\tau + F(d'(x', x'')) = \tau + F(d'(fx', gx'')) \leq F(M_2(x', x''))$$

Where,

$$M_2(x', x'') = \max\left\{d'(x', x''), d'(x', fx''), d'(x', gx''), \frac{d'(x', gx'') + d'(x', fx'')}{4}\right\}$$
$$= d'(x', x'')$$

Therefore from the above inequality we have

$$\tau + F(d'(x', x'')) \leq F(d'(x', x''))$$

— a contradiction.

Thus

$$d'(x', x'') = 0$$
$$\Rightarrow x' = x''.$$

## 4. APPLICATIONS

The following results are the direct applications of the above theorem;

**Theorem 4.1** *Suppose $(X, d')$ be a 0-complete metric like space and $f, g: X \to X$ be two functions such that whenever $d'(fx, gy) > 0$ there exists $\tau > 0$ such that*

$$\tau + \log(d'(fx, gy)) \leq \log(M_2(x, y)) \qquad \text{Equation 5}$$

Then $f$ and $g$ will have a unique common fixed point in X.

*Proof.* Considering $F(t) = \log t \in \mathcal{F}$ in Equation 1 we can have the following results.

**Theorem 4.2** *Suppose* $(X, d')$ *be a 0-complete metric like space and* $f, g: X \to X$ *be two functions such that whenever* $d'(fx, gy) > 0$ *there exists* $\tau > 0$ *such that*

$$\tau + d'(fx, gy) + \log(d'(fx, gy)) \leq M_2(x, y) + \log(M_2(x, y)) \qquad \text{Equation 6}$$

Then $f$ and $g$ will have a unique common fixed point in X.

*Proof.* Considering $F(t) = t + \log t \in \mathcal{F}$ in equation Equation 1 we can have the following results.

## 5. AUTHORS CONTRIBUTION

Conceptualization: M.Mitra, J.D. Bhutia.

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Formal Analysis: M.Mitra, J.D. Bhutia, K. Tiwary.

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## 6. CONSENT OF STATEMENT

All authors have read and agreed to publish this manuscript.

## CONFLICT OF INTERESTS

None.

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