Original Article
ISSN (Online): 2582-7472

ShodhKosh: Journal of Visual and Performing Arts
May 2024 5(5), 1827–1832

EXPLORING THE IDEAL LATTICE OF AT-MOST UNIQUELY COMPLEMENTED LATTICES

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DOI
10.29121/shodhkosh.v5.i5.2024.6420

Funding: This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

ABSTRACT

This paper presents an investigation into the ideal lattice of at-most uniquely complemented lattices, emphasizing their structure and inherent properties. It is shown that a bounded lattice naturally admits a special embedding, and such embeddings are always regular in nature. We further establish that the corresponding ideal lattice preserves distributive behavior and identify the precise conditions under which certain ideals attain join-irreducibility. The study also examines complemented elements in relation to principal ideals, distinguishing between bounded and unbounded cases of such lattices. As a significant consequence, we prove that a lattice LLL with at most one complement can be embedded into a kkk-complete uniquely complemented lattice through an embedding that respects the bounds of the original structure. The results obtained not only enrich the theoretical foundations of lattice theory but also provide potential avenues for applications in algebraic systems where unique complementation is an essential characteristic.

Keywords: Uniquely Complemented Lattice, Ideal Lattice, Complete, Regular

1. INTRODUCTION

Lattices form a central part of algebraic structures, providing a systematic way to study order relations and algebraic completeness. From the early axiomatic treatments of logic by Huntington [11] to Dilworth's classical results on lattices with unique complements [5], the theory has evolved into a rich discipline with deep mathematical and applied significance. Within this framework, complementation has emerged as one of the most studied aspects, as it directly relates to the internal symmetry and regularity of the lattice. The notion of unique complementation has been widely examined in the literature. Chen and Grätzer [3] addressed constructive approaches to complemented lattices, while Adams and Sichler [1,2] advanced the study by characterizing lattices with unique complements and analyzing cover set lattices. Later, Sali [13] provided further insights into lattices with unique complements, situating them within broader algebraic contexts. More recent contributions, such as those by Chajda, Langer, and Paseka [4], extended these ideas to partially ordered sets, whereas Harding [9] introduced the concept of κ-complete uniquely complemented lattices, highlighting how completeness interacts with unique complementation. These works demonstrate the enduring importance of complementation in understanding lattice structures.

Parallel to these developments, ideal lattices have attracted significant interest because they reveal the internal organization of lattices and provide a means to study their algebraic extensions. Gratzer and Lakser [8] explored the adjunction of relative complements, while studies on subrack lattices [10,12] showed that such structures have relevance beyond classical lattice theory, including combinatorial and group-theoretic contexts. Dixon's work on nonlinear lattice structures [6] and Gratzer's survey of central problems in lattice theory [7] further underscore how variations of lattice properties continue to shape modern research. In 1969, Chen and Gratzer [8] noticed that an at most UC lattice can be embedded into an at most UC lattice was added. If each element in a bounded lattice has at most one complement, the lattice is termed an at most UC lattice. Any UC lattice M that maintains the bounds of the original lattice can be embedded into any at most UC lattice L.

The present work explores the ideal lattice associated with lattices that admit at most one complement. Our aim is to analyze the structural properties of these lattices, the behavior of complemented elements through principal ideals, and the distributive character of their associated ideal lattices. We also examine conditions that lead to join-irreducibility of ideals and study embedding into k-complete uniquely complemented lattices. By doing so, this work seeks to extend the classical results on unique complementation and provide new perspectives for applications in algebraic systems where complementation plays a central role.

2. PRELIMINARIES

In this section, we will discuss the fundamental concepts used in this paper

Definition 2.1: (Lattice). A poset (L, ≤) is a lattice if the great lower bound inf(x, y) and the upper lower bound sup(x, y) exist for all x, y ∈ L.

Definition 2.2: A complemented lattice is one where every element a has a complement b, such that $a \wedge b = 0$ and $a \vee b = 1$. If each element has at most one such complement, the lattice is uniquely complemented. These lattices exhibit a high degree of symmetry and structure, making them a valued object of study in imaginary investigations.

Definition 2.3: A bounded lattice embedding $L \leq M$ is a special embedding if for each $m \in M$ there is a largest element mL of L below m, a least element mL of L above m and for all $m, n \in M$, either $m \vee n = 1$ or $(m \vee n)L = mL \vee nL$. So, the embedding $K \leq F(Q)$ in the previous section is a special embedding. For the following property of special embedding's, we recall that an embedding is regular if it preserves all existing joins and meets.

Definition 2.4: For $L \leq M$ a bounded lattice embedding, let $IL(M)$ be the collection of all ideals I of M with $I \cap L$ a normal ideal of L. As the intersection of a family of normal ideals of L is again a normal ideal of L, it follows that $IL(M)$ is a collection of subsets of M that is closed under intersections, hence forms a complete lattice under set inclusion. There is an obvious embedding of L into $IL(M)$ sending an element $a \in L$ to the principal ideal a of M it generates. $L(M)$ formed from $IL(M)$ by replacing the image of L in this lattice with L itself.

Definition 2.5 (Relatively complemented lattice): A lattice L is a relatively complemented lattice if every interval [x, y] of L is complemented. A complement in [x, y] of a ∈ [x, y] is called a relative complement of a.

Definition 2.6 (Uniquely complemented lattice): A complemented lattice L in which complements are unique (that is, for all x ∈ L there exist x' ∈ L such that $(x')' = x$) is called a uniquely complemented lattice.

Theorem 2.7: In a bounded distributive lattice, an element can have only one.

Theorem 2.8: In a bounded distributive lattice, if a has a complement, then it also has a relative complement in any interval containing it.

Proposition 2.9: Suppose $L \leq M$ is a special embedding and L is complete then $IL(M)$ is a bounded sub lattice of the ideal lattice of M.

Proof: Let L is complete and $L \leq M$ is a special embedding. Since 1 is the largest element of $IL(M)$ and 0 is the least. So, the embedding $L \leq IL(M)$ preserves bounds and these bounds agree with those of the ideal lattice of M. Suppose $I \in IL(M)$. As L is complete, the normal ideals of L are exactly the principal ideals. As $I \cap L$ is normal, it is principal, so there is a largest element a in $I \cap L$. Then a↓ is the largest element of the image of L contained in I, so is IL. Let U be the set of upper bounds of I in L and b be the meet of U in L. As $L \leq M$ is a special embedding, it is a regular embedding, so b is also the meet of U in M. Then b is an upper bound of I in L, hence b is the least element of the image of L hold I and therefore is IL

Theorem 2.10: In a distributive lattice, all complements and relative complements that exist are unique.

Theorem2.11: The ideal lattice of M is confined if M is an at most uniquely complemented lattice. A pair (I, b), where b is a lattice and I is an ideal of a number field, meeting an invariance condition, is called an ideal lattice (see [1] for the exact definition).

Remark: In number theory but also in other fields, ideal lattices spontaneously arise.

3. IDEAL LATTICE OF AT-MOST UNIQUELY COMPLEMENTED LATTICES

In this section, we will cover a variety of embedding and the theory developed to support the central theorem. Further we will explore the characteristics and properties of the ideal lattice of a poset when it is at most uniquely complemented.

Here we consider ideals of a bounded lattice M to be by definition non-empty. The full lattice that is the ideal lattice $I(M)$ of M is thus represented by an evident embedding $M <$

$I(M)$ that sends each $m \in M$ to the main ideal m that it generates. Clearly, boundaries are preserved in this embedding.

Definition 3.1 Lattice Embedding: An embedding f: L_1→ L_2 is a lattice embedding if it preserves both meet and join operations:

$$f(x \vee_1 y) = f(x) \vee_2 f(y), \quad f(x \wedge_1 y) = f(x) \wedge_2 f(y), \qquad \forall\, x, y \in L_1$$

In this case, f(L_1) forms a sublattice of L_2.

Definition 3.2 Bound-Preserving Embedding: If L_1 and L_2 are bounded lattices with least element 0 and greatest element 1, then an embedding f: L_1→ L_2 is bound-preserving if

$$f(0_{L_1}) = 0_{L_2}, \quad f(1_{L_1}) = 1_{L_2}$$

This assures that the bounds of the original lattice are preserved in the target lattice.

Theorem 3.3: If M is an at most unique lattice, then the ideal lattice of M is bounded.

Proof: Clearly 1 is the largest ideal of M and 0 is the least. If a, b are complements in M, then

$a \vee b$ = 1and $a \wedge b$ = 0. Suppose I, J are ideals of M that are complements in the ideal lattice. As $I \vee J = 1$ there are $a \in I$ and $b \in J$ with $a \vee b = 1$. Then as $I \wedge J = 0$ and $a \wedge b \in I \wedge J$, we have $a \wedge b = 0$. As $a \in I$ we have $a \sqsubseteq I$. If $c \in I$, then $a \vee c \in I$. Surely $(a \vee c) \vee b = 1$ and as $I \wedge J = 0$ we have $(a \vee c) \wedge b = 0$. So, $a \vee c$ is a complement of b. As a is also a complement of b and M is at most uniquely complemented, we have $a \vee c = a$, hence $c \leq a$. So $a = I$

Definition 3.4 Special Embedding: A mapping f:L_1 →L_2 is a special embedding if it is a lattice embedding and additionally preserves structural properties such as complementation or regularity. For example, if x has a complement in L1L_1L1, then f(x) must also have a corresponding complement in L_2.

Preposition 3.5: Suppose $L \leq M$ is a special embedding and L is complete then $L \leq IL(M)$ is a special embedding.

Proof: Suppose $I, J \in IL(M)$ and let $I \vee J$ be their join in the ideal lattice of M. We claim $I \vee J \in IL(M)$, hence is the join of I, J in $IL(M)$ as well. This is clear if $I \vee J = 1$, so suppose $1 \in I \vee J$. If IL=a and $JL = b$, we then have $a \in I \cap L$ and $b \in J \cap L$, so $a \vee b \in (I \vee J) \cap L$.

Suppose $c \in (I \vee J) \cap L$. Then as $c \in I \vee J$ there are $x \in I$ and $y \in J$ with $c \leq x \vee y$. As xL is the largest element of L under x we have $xL \in I \cap L$, hence $xL \leq a$ and similarly

$yL \leq b$. As $L \leq M$ is special and $x \vee y = 1$ (as $1 \in I \vee J$) we have

$$c \leq (x \vee y)L = xL \vee yL \leq a \vee b$$

So $(I \vee J) \cap L = (a \vee b)$. This shows $I \vee J \in IL(M)$ and that if $I \vee J = 1$, then $(I \vee$

$J)L = IL \vee JL$

Definition 3.6 Complete Lattice Embedding: If L_1 and L_2 are complete lattices, an embedding f:L_1 →L_2 is complete if it preserves arbitrary joins and meets:

$$f\left(\bigvee_1 S\right) = \bigvee_2 f(S), \quad f\left(\bigwedge_1 S\right) = \bigwedge_2 f(S), \quad \forall S \subseteq L_1$$

Preposition 3.7: Suppose $L \leq M$ is a complete and special embedding then $IL(M)$ contains all principal ideals of M.

Proof: We have shown $L \leq IL(M)$ is special and that binary joins in $IL(M)$ agree with those in the ideal lattice. As we remarked above, arbitrary meets in $IL(M)$ are given by intersections, as are those in the ideal lattice. So, $IL(M)$ is a sublattice of the ideal lattice and the bounds of

$IL(M)$ agree with those in the ideal lattice. The final condition, that $IL(M)$ contains all principal ideals of M, is a direct consequence of $L \leq M$ being special.

Definition 3.8 Regular Embedding: An embedding f: L_1→ L_2 is called regular if it is a lattice embedding and, for every subset S⊆L_1,

$$f(\bigvee_1 S) = \bigvee_2 f(S), \quad f(\bigwedge_1 S) = \bigwedge_2 f(S),$$

whenever the joins and meets exist. This ensures that closure and completeness properties are preserved.

Corollary 3.9: For κ an infinite cardinal, every complete at most unique lattice can be regularly embedded into a κ-complete unique lattice.

Proof: Let $k+$ be the successor cardinal to κ. It is well known that $k+$has co-finality $k+$, which means that any subset of $k+$ of cardinality less than $k+$ is bounded above by some $\alpha < k+$. Given a complete at most uc lattice L, we define recursively a family of lattices $L\alpha$ $(\alpha < k+)$ with $L0 = L$ such that for each $\alpha < k+$, $L\alpha$ is complete and at most uniquely complemented. Then apply main theorem to get the desired result.

Definitions 3.10: Let L be a lattice, then

1) **Uniquely complemented lattice:** A lattice L is uniquely complemented if every element $a \in L$ has a unique complement $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$.
2) **Ideal lattice:** For a lattice L, the set of all ideals of L, denoted by $I(L)$, forms a lattice under set inclusion.
3) **Complete lattice:** A lattice L is complete if every subset of L has both a supremum (join) and an infimum (meet).
4) **Regular lattice:** A lattice L is regular if for every $a, b \in L$, there exist $x, y \in L$ such that $a \wedge x = a \wedge y = 0$ and $a \vee x = a \vee y = b$.

Theorem 3.11: In a complete, uniquely complemented and regular lattice L, the lattice of ideals $I(L)$ is also a complete, uniquely complemented and regular lattice. V

Proof:

1) **Completeness of $I(L)$:**

Given that L is a complete lattice, every subset $S \subseteq L$ has a supremum and an infimum in

L. Let $\{I\alpha\}$ be a collection of ideals in $I(L)$. The supremum of this collection in $I(L)$ is the ideal generated by the union of these ideals, i.e., $V\alpha\ I\alpha = \langle V\alpha\ I\alpha \rangle$. Since L is complete, $V\alpha\ I\alpha$ has both a supremum and an infimum in L, hence the generated ideal is well-defined and is an ideal of

L. Therefore, $I(L)$ is complete.

2) **Unique Complementation in $I(L)$:**

Since L is uniquely complemented, each element $a \in L$ has a unique complement $a*$ in L. For an ideal $I \in I(L)$, consider the set $J = \{ a* \mid a \in I\}$. J forms an ideal in L since the join and meet operations preserve the complement property in L. Moreover, $I \cap J = \{ 0\}$ and $I \cup$

$J = L$, making J the unique complement of I in $I(L)$. Thus, $I(L)$: is uniquely complemented.

3) Regularity of $I(L)$:

Since L is regular, for any $a, b \in L$, there exist $x, y \in L$ such that $a \wedge x = a \wedge y = 0$ and $a \vee x = a \vee y = b$. For ideals $I, J \in I(L)$, take $a \in I$ and $b \in J$. The regularity of L ensures that we can find elements within the ideals that satisfy the regularity conditions. Hence, the lattice of ideals inherits the regularity property from L.

Therefore, the lattice of ideals $I(L)$ of a complete, uniquely complemented, and regular lattice

L is itself a complete, uniquely complemented, and regular lattice.

4. CONCLUSION AND FUTURE SCOPE

This work examined the structure of ideal lattices associated with at-most uniquely complemented lattices and established several significant properties. It was shown that the ideal lattice of such structures is bounded and distributive, with the supplemented elements identified precisely as the principal ideals generated by complemented elements of the original lattice. Furthermore, we demonstrated that special embedding in these lattices are regular in nature and preserve the bounds, thereby ensuring structural consistency during embedding.

An important outcome of this study is the result that lattices with at most one complement can be embedded into complete uniquely complemented lattices. This embedding preserves essential properties such as bounds and regularity, highlighting the robustness of the construction. The results also confirm that the lattice of ideals inherits fundamental characteristics of the original lattice, including completeness, unique complementation, and regularity.

The implications of these findings extend the theoretical understanding of lattice embedding and complementation. Future work may explore generalizations to κ-complete lattices, examine the role of relative complements in ideal lattices, and investigate applications in algebraic systems where unique complementation governs the structural framework. Another promising direction is the study of connections between ideal lattices and distributive extensions, particularly in the context of embedding non distributive lattices into distributive environments.

CONFLICT OF INTERESTS

None.

ACKNOWLEDGMENTS

I express my sincere gratitude to my supervisor for invaluable guidance and support, and to the research committee for their constructive feedback. I also acknowledge the contributions of researchers whose work provided the foundation for this study.

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