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SOLVING FPDES & INERTIAL DISCRETE TRANSFER FUNCTION USING ATANGANA-BLAENAU OPERATOR

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ABSTRACT

In this work, the natural homotopy permutation technique is discussed as follows: In order to make this endeavor scientifically valuable, the Atangana-Baleanu operator in the Reimann sense was applied with fractional differential equations to solve them using this method. Definitions and characteristics related to this study are also given, and the algorithm of the methodology is also examined. These equations' approximate solutions were eventually discovered, and the method worked well for resolving this kind of fractional problem.

Fractional order, discrete transfer function model of an elementary inertial plant is proposed. The model uses Atangana-Baleanu operator. The discrete transfer function's convergence and stability are examined. Simulations extend theoretical results. The suggested discrete, approximated model has a minimal numerical complexity and is correct. When modeling various physical phenomena, such as heat processes, it might be helpful.

Keywords: Natural Transform, Homotopy Permutation Method, Fractional Order Transfer Function, Grünwald-Letnikov Definition, Convergence

1. INTRODUCTION

Fractional calculus and its applications in mathematics, as well as in many other sciences including physics, thermodynamics, engineering, economics, etc., have attracted the attention of researchers in recent decades. There are numerous uses for fractional calculus in the fields of probability, statistics, electrochemistry, and electrical engineering. Furthermore, many cosmic phenomena that conventional differential equations are unable to describe can be described by fractional differential equations [1–5].

In recent decades, researchers have been interested in studying fractional calculus and its applications in a wide range of disciplines, including economics, physics, thermodynamics, geology, and mathematics. There are numerous uses for fractional calculus in statistics, probability, electrochemistry, and electrical engineering. Additionally, a variety of cosmic events that normal differential equations are unable to describe may be described by fractional differential equations [5–10].

In this research, we use the fractional operator Atangana-Baleanu-Reimann to solve fractional differential equations using the natural homotopy permutation technique. The paper is organized as follows: Section 2 presents the fundamental definitions of calculus and fractional integration; Section 3 analyzes the methods employed; Section 4 provides numerous examples that demonstrate the efficacy of the suggested method; and Section 5 concludes.

In the publication [14], Atangana and Baleanu proposed a novel fractional operator with a nonsingular kernel. Papers [23,25] investigate the approximations of the Atangana Bealeanu operator (AB operator).

Papers [19] or [20], for example, have intriguing sets of data demonstrating the application of the AB operator in modeling many physical, biological, and social processes. There are blood alcohol models, logistic equation models, and population growth models.

The following publications provide some recent findings demonstrating the application of the AB operator: The use of the AB operator in nonlinear fractional differential equations is covered in the work [31], the application of the AB operator in advection-dispersion equations is shown in the paper [13], and [15] examines the modeling of COVID-19 dynamics in India. For instance, [30] and [24,26] examined the simulation of heat transmission using the AB operator.

All known models that use the AB operator have the form of a state equation, which is a characteristic. There is currently no proposed transfer function model that makes use of this operator.

In this research, two iterations of a novel AB operator-based fractional order transfer function model are proposed. The time-continuous model and its step response are shown first.

Several actual physical occurrences can be modeled using the recently suggested model.

2. PRELIMINARY

Definition 1 [7,8] Let $\mathscr{v} \in H^1(\varepsilon_1, \varepsilon_2), \varepsilon_1 > \varepsilon_2$, the Reimann-sense Atangana-Baleanu operator for $< \delta <$ is,

$${}^{ABR}_{\ a}D_t^{\delta}\mathscr{v}(t) = \frac{B(\delta)}{1-\delta}\frac{d}{dt}\int_0^t E_\delta\left(-\frac{\delta(t-T)^\delta}{1-\delta}\right)\mathscr{v}(T)dT, t \geq 0, \quad (1)$$

Where B(δ) is the normalizing function such that

B (0) = B (1) =1.

Definition 2 [9,10] The set of functions is where the natural transform is defined.

$$A = \left\{\mathscr{v}(t)|\exists \mathcal{M}, \tau_1, \tau_2 > 0, |\mathscr{v}(t)| < \mathcal{M}e^{\frac{t}{\tau_j}}, if\ t \in (-1)j \times [0, \infty)\right\} \quad (2)$$

Using the formula below,

$$N[\mathscr{v}(t)] = R(u,s) = \int_0^\infty \mathscr{v}\,(ut)e^{-st}dt, u \in (\tau_1, \tau_2).$$

Definition 3 [11] The definition of a function's inverse natural transform is

$$N^{-1}[R(u,s)] = \mathscr{v}(t) = \frac{1}{2i\pi}\int_{p-\infty}^{p+\infty} e^{\frac{st}{u}} R(u,s)dt, u, s > 0 \quad (3)$$

where p is a real constant and s and u are natural transform variables δ.

The following connection [12] allows the natural transform to provide the Laplace transform.

$$R(u,s) = \frac{1}{u}\int_0^\infty e^{-st/u} f(t)dt = \frac{1}{u}F\left(\frac{s}{u}\right).$$

Equation (3) and [12] provide us with this relationship,

$$N\left({}^{ABR}_{\ a}D_t^{\delta} f(\tau)\right) = \frac{B(\delta)}{1-\delta+\delta\left(\frac{u}{s}\right)^\delta}R(u,s) \quad (4)$$

2.1. BASICS OF FRACTIONAL CALCULUS

Many books contain basic concepts from fractional calculus, such as [18,21,28] or [29]. Only a few definitions required to present the primary findings will be remembered here.

It is necessary to first provide the fractional-order, integro-differential operator (see, for example, [18, 22, 29]). It is as follows:

Definition 4. The basic operator for fractional order) The following is the definition of the fractional-order integro-differential operator:

$$ {}_{t_s}D_{t_f}^{\alpha} f(t) = \begin{cases} \frac{d^{\alpha} f(t)}{dt^{\alpha}} & \alpha > 0, \\ f(t) & \alpha = 0, \\ \int_{t_s}^{t_f} f(\tau)(d\tau)^{\alpha} & \alpha < 0, \end{cases} \quad (5) $$

where $\alpha \in R$ indicates the operation's non-integer order, and t_s and t_f indicate the time constraints for operator computation. Next, review the Gamma Euler function (see to [22] for example):

Definition 5. (The Gamma function)

$$ \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad (6) $$

A non-integer order generalization of the exponential function e^λt, the Mittag-Leffler function is essential to solving the FO state equation. The definition of the one-parameter Mittag-Leffler function is as follows:

Definition 6. (The Mittag-Leffler function with a single parameter)

$$ E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+1)}. \quad (7) $$

The following is the definition of the two-parameter Mittag-Leffler function

Definition 7. (The Mittag-Leffler function's two arguments)

$$ E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+\beta)} \quad (8) $$

For $\beta = 1$ Function (8) with two parameters becomes function (7) with one parameter.

Different definitions of the fractional-order, integro-differential operator have been proposed by Riemann and Liouville (RL definition), Caputo (C definition), and Grünvald and Letnikov (GL definition). The definitions of C and GL will be applied in the subsequent analysis. Below, they are provided [17,27].

Definition 8. (The FO operator as defined by Caputo)

$$ {}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^{\infty} \frac{f^{(N)}(\tau)}{(t-\tau)^{\alpha+1-N}} d\tau \quad (9) $$

where N-1<α<N indicates Γ and the non-integer order of operation.is the whole Gamma function that (6) expresses. The Laplace transform for the Caputo operator can be defined as follows (see [21] for example):

Definition 9. (The Caputo operator's Laplace transform)

$$ \mathcal{L}\left({}_0^C D_t^{\alpha} f(t)\right) = s^{\alpha} F(s), \qquad \alpha < 0, $$

$$ \mathcal{L}\left({}_0^C D_t^{\alpha} f(t)\right) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} \; {}_0D_t^k f(0), \quad (10) $$

$$ \alpha > 0, n-1 < \alpha \le n \in N. $$

Thus, the following is the expression for the inverse Laplace transform for a non-integer order function ([22]):

$$\mathcal{L}^{-1}[s^{\alpha}F(s)] = {}_0D_t^{\alpha}f(t) + \sum_{k=0}^{n-1}\frac{t^{k-1}}{\Gamma(k-\alpha+1)}f^{(k)}(0^+) \quad (11)$$

$$n-1 < \alpha, \quad n \in \mathbb{Z}$$

The following is the definition of the GL derivative along time from function [17, 27]:

Definition 10. (The Grünwald-Letnikov definition)

$${}_0^{GL}D_t^{\alpha}g(t) = \lim_{h\to 0} h^{-\alpha}\sum_{l=0}^{\left[\frac{t}{h}\right]}(-1)^l\binom{\alpha}{l}g(t-lh). \quad (12)$$

In (12) "$0.0 < \alpha \leq 1.0$ is the fractional order along the time, h is the sample time, [.] is the nearest integer value, $\binom{\alpha}{l}$ is the binomial coefficient:

$$\binom{\alpha}{l} = \begin{cases} 1, & l = 0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha-l+1)}{l!}, & l > 0 \end{cases} \quad (13)$$

3. ANALYSIS OF THE PROPOSED METHOD

Suppose that fractional partial differential equation with Atangana-Baleanu- Reimann operator,

$${}_a^{ABR}D_t^{\delta}v(x,t) + L[v(x,t)] + M[v(x,t)] = g(x,t),$$

With initial condition v(x,0)=v_0 (x), where ABRD_τ^δ is the Atangana - Baleanu - Reimann operator, L is a linear operator, M is a nonlinear operator and g is a source term.

Applying the natural transform of Atangana-Baleanu- Reimann operator subject to the given initial condition,

$$\frac{B(\delta)}{1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}}v(u,s) = N(g(x,t) - L[v(x,t)] - M[v(x,t)]),$$

By substituting initial condition of natural transform of Atangana-Baleanu- Reimann operator

$$v = -\frac{1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}}{B(\delta)}N(L[v] + M[v] - g), \quad (14)$$

Applying the inverse of the natural transform to both sides of the Eq. (14),

$$v = N^{-1}\left(\frac{1-\delta+\delta\left(\frac{u}{8}\right)^{\delta}}{B(\delta)}N(g)\right) - N^{-1}\left(\frac{1-\delta+\delta\left(\frac{u}{8}\right)^{\delta}}{B(\delta)}N(L[v] + M[v])\right), \quad (15)$$

By applying homotopy permutation method,

$$v(x,t) = \sum_{n=0}^{\infty}p^n\,v_n(x,t), N[u(x,t)] = \sum_{n=0}^{\infty}p^n\,H_n(v), \quad (16)$$

Were

$$H_n(v_1, v_2, v_3, \ldots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{n}p^i\,v_i(x,t)\right)\right]_{p=0} n = 0,1,2, \ldots. \quad (17)$$

Substituting Eq. (16) into Eq. (15) gives us the result that,

$$\sum_{n=0}^{\infty}p^n\,v_n(x,t) = G(x,t) - p\left(N^{-1}\left(\frac{1-\delta+\delta\left(\frac{u}{8}\right)^{\delta}}{B(\delta)}N\left(\sum_{n=0}^{\infty}p^n\,L[v_n] + \sum_{n=0}^{\infty}p^nH_n(v)\right)\right)\right) (18)$$

Were

$$G(x,t) = v(x,0) + N^{-1}\left(\frac{1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}}{B(\delta)}N(g)\right) \quad (19)$$

By comparing both sides of the equation, the following result is obtained,

$$p^0: v_0(x,t) = G(x,t),$$

$$p^1: v_1(x,t) = -N^{-1}\left(\frac{1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}}{B(\delta)} N\big(L[v_0] + H_0(v)\big)\right), \quad (20)$$

$$p^n: v_n(x,t) = -N^{-1}\left(\frac{1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}}{B(\delta)} N\big(L[v_{n-1}] + H_{n-1}(v)\big)\right), \quad (21)$$

Using the parameter p, we expand the solution in the following form,

$$v(x,t) = \sum_{n=0}^{\infty} p^n\, v_n(x,t) \quad (22)$$

Setting p=1 results in the solution of Eq. (22)

$$v(x,t) = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n\, v_n(x,t) = \sum_{n=0}^{\infty} v_n\,(x,t).. \quad (23)$$

3.1. ELEMENTARY FO TRANSFER FUNCTION

The elementary, scalar input-output differential equation using elementary fractional operator (5) takes the following form:

$$T_{\alpha 0} D_t^{\alpha} y(t) = -y(t) + u(t). \quad (24)$$

where T_α is the time constant, u(t) is the control signal and y(t) is the output.

Assume homogenous initial condition. Applying (10) in (24) gives the elementary, fractional order transfer function:

$$G_C(s) = \frac{1}{T_\alpha s^\alpha + 1}. \quad (25)$$

For this transfer function its impulse and step responses are as beneath (see e.g. [17, p. 11]):

$$g_C(t) = \frac{t^{\alpha-1}}{T_\alpha} E_\alpha\left(-\frac{t^\alpha}{T_\alpha}\right),$$
$$y_C(t) = 1(t) - E_\alpha\left(-\frac{t^\alpha}{T_\alpha}\right). \quad (26)$$

In (26) E_α. (.) is the one parameter Mittag-Leffler function (7)

3.2. THE FOBD APPROXIMATION

The GL definition is the limit case for h→0,Δx→0 of the FOBD, commonly employed in discrete FO calculations (see e.g. [28, p. 68]).

Definition 11. (The Fractional Order Backward Difference along the time FOBDT)

$$\Delta^\alpha g(t) = \frac{1}{h^\alpha} \sum_{l=0}^{L} (-1)^l \binom{\alpha}{l} g(t - lh) \quad (27)$$

In (27) L denotes a memory length necessary to correct approximation of a non integer order operator. Unfortunately, good accuracy of approximation requires to use a long memory L which can make implementation difficult.

Denote coefficients $(-1)^l \binom{\alpha}{l}$ by d_l :

$$d_l = (-1)^l \binom{\alpha}{l} \quad (28)$$

The coefficients (28) can be also computed using the following, equivalent, recursive formula (e.g. [17, p. 12]), useful in numerical calculations:

d_0=1

$$d_l = \left(1 - \frac{1+\alpha}{l}\right) d_{l-1}, \quad l = 1, \ldots, L \quad (29)$$

In [16] it is given that:

$$\sum_{l=1}^{\infty} d_l = 1 - \alpha \qquad \sum_{l=0}^{\infty} d_l = 0 \qquad (30)$$

Using (28) the operator (27) can be expressed in shorter form:

$$\Delta^{\alpha} g(t) = \frac{1}{h^{\alpha}} \sum_{l=0}^{L} d_l \, g(t - lh) \qquad (31)$$

and consequently, its discrete transfer function G_FOBD $G_{FOBD}(z^{-1})$ takes the following form:

$$G_{FOBD}(z^{-1}) = \frac{1}{h^{\alpha}} \sum_{l=0}^{L} d_l \, z^{-l} \qquad (32)$$

3.3. DISCRETE SYSTEMS: SELECTED RESULTS

Let recall two theorems from theory of discrete time dynamic systems, necessary to present of main results: there are Final Value Theorem (FVT) and necessary condition of the asymptotic stability of a system described by a discrete transfer function G^(+(z)).

Theorem 1. (Final Value Theorem for discrete time) Let g(k) is a discrete function of time, defined in k time instants and G(z) is its z-transform. Assume that G^(+(z)) :

1) has no poles outside the unit circle,
2) has maximally one pole on the unit circle: z=1,then:

$$\lim_{k \to \infty} g(k) = \lim_{z \to 1} (z - 1) G(z) \qquad (33)$$

Theorem 2. (Necessary condition of the asymptotic stability of the discrete polynomial)

Consider the characteristic polynomial of a discrete system: w(z)=a_N z^N+⋯+a_1 z+a_0.

The necessary condition of its asymptotic stability is as follows:

$$w(1) > 0 \wedge (-1)^N w(-1) > 0 \wedge |a_0| < a_N \qquad (34)$$

3.4. THE ATANGANA-BALEANU FRACTIONAL OPERATOR

The fractional order derivative Atangana-Baleanu operator is obtained via replacing the exponential kernel in the Caputo-Fabrizio (CF) operator by the Mittag-Leffler kernel. It is defined using the C or RL definition of fractional order derivative. Using these definitions we obtain the Atangana-Baleanu-Caputo (ABC) or Atangana-Baleanu-Riemann (ABR)operator respectively [14]:

Definition 12. (The Atangana-Baleanu-Caputo (ABC) operator)

$${}^{ABR}_{a}D_t^{\alpha}\big(f(t)\big) = M(\alpha) \int_a^t f'(x) E_{\alpha}\left(-\alpha \frac{(t-x)^{\alpha}}{1-\alpha}\right) dx, \qquad (35)$$

where E_α.is the oneparameterMittag-Lefflerfunction,M_α is the normalization function equal:

$$M(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)} \qquad (36)$$

In (36) Γ (.) is the Gamma function.

Definition 13. (The Atangana-Baleanu-Riemann (ABR) operator)

$${}^{ABR}_{a}D_t^{\alpha}\big(f(t)\big) = M(\alpha) \frac{d}{dt} \int_a^t f(x) E_{\alpha}\left(-\alpha \frac{(t-x)^{\alpha}}{1-\alpha}\right) dx. \qquad (37)$$

where E_α(.) is the one parameter Mittag-Leffler function, $M(\alpha)$ is the normal ization function expressed by (36), Γ(.) is the Gamma function.

The Laplace transforms for the ABC and ABR derivatives are as follows:

Definition 14. (The Laplace transform of the ABC operator)

$$\mathcal{L}\{{}^{ABR}_{a}D_t^{\alpha}(f(t))\}(s) = \frac{M(\alpha)}{1-\alpha}\frac{s^{\alpha}\{f(t)\}(s)-s^{\alpha-1}f(0)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}. \quad (38)$$

Definition 15. (The Laplace transform of the ABR operator)

$$\mathcal{L}\{{}^{ABR}_{a}D_t^{\alpha}(f(t))\}(s) = \frac{M(\alpha)}{1-\alpha}\frac{s^{\alpha}\{f(t)\}(s)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}. \quad (39)$$

For the homogenous initial condition: f(0)=0 both Laplace transforms are equal:

$$\mathcal{L}\{{}^{ABR}_{a}D_t^{\alpha}(f(t))\}(s) = L\{ABC_aD_t^{\alpha}(f(t))\}(s). \quad (40)$$

In further considerations it will be used the common notation AB to denote this operator in both versions, becuase the initial conditions are equal zero during analysis of a transfer function. To simplify, introduce the following short notation:

$$\mathcal{L}\{{}^{ABR}_{a}D_t^{\alpha}(f(t))\}(s) = \frac{b_{\alpha}s^{\alpha}}{s^{\alpha}+a_{\alpha}} \quad (41)$$

where:

$$a_{\alpha} = \frac{\alpha}{1-\alpha}, \quad (42)$$

$$b_{\alpha} = \frac{M(\alpha)}{1-\alpha}. \quad (43)$$

The form a_α and b_α require to assume that "0.0"≤α<"1.0" .

4. IMPLEMENTATIONS

Example 1: Let us consider the following nonlinear equation with the Atangana-Baleanu- Reimann sense.

$${}^{ABR}_{a}D_t^{\delta}\varphi(\mu,\tau) + 2\varphi_{\mu} + \varphi_{\mu\mu} = 0, ,0 < \delta \leq 1$$

Subject to the initial condition φ(μ,0)="sin" (μ).By using the natural transform to both sides of Atangana-Baleanu-Reimann sense,

$$N[{}^{ABR}_{a}D_t^{\delta}\varphi(\mu,\tau) + 2\varphi_{\mu}(\mu,\tau) + \varphi_{\mu\mu}(\mu,\tau) = 0] \quad (43)$$

By using the inverse natural transform to both sides of (43) and the initial condition,

$$\varphi(\mu,\tau) = -N^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{\delta}\right)^{\delta}\right)N[2\varphi_{\mu}+\varphi_{\mu\mu}]\right] \quad (44)$$

By applying homotopy permutation method on Eq. (44),

$$\sum_{n=0}^{\infty} P^n\,\varphi_n = -pN^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right)N\left[2\sum_{n=0}^{\infty}P^n\,\varphi_{n\mu} + \sum_{n=0}^{\infty}p^n\,\varphi_{n\mu\mu}\right]\right] \quad (45)$$

By comparing both sides of the Eq. (45), the following result is obtained,

$$P^0: \varphi_0 = \sin(\mu),$$

$$P^1: \varphi_1 = -N^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right)N[2P^0\varphi_{0\mu} + P^0\varphi_{0\mu\mu}]\right]$$

$$P^2: \varphi_2 = -N^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right)N[2P^1\varphi_{1\mu} + P^1\varphi_{1\mu\mu}]\right] \quad (47)$$

By the above algorithms,

$$\varphi_0 = \sin(\mu)$$

$$\varphi_1 = -\cos(\mu)\left(1-\delta+\delta\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right)$$

$$\varphi_2 = \sin(\mu)\left((1-2\delta+\delta^2)+(2\delta-2\delta^2)\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\right)$$

and so on. Therefore, the series solution φ(μ,τ) of the Atangana-Baleanu- Reimann sense is given by,

$$\varphi(\mu,\tau) = \sin(\mu)-\cos(\mu)\left(1-\delta+\delta\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right)+\sin(\mu)\begin{pmatrix}(1-2\delta+\delta^2)\\ +(2\delta-2\delta^2)\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\end{pmatrix}-\cdots.$$

(48)

If we put δ→1 in Eq. (48), we get the approximate and exact solution,

$$\varphi(\mu,\tau) = \sin(\mu)-\tau\cos(\mu)+\frac{\tau^2}{2!}\sin(\mu)-\cdots = \sin(\mu-\tau)$$

Example 2 Let us consider the following nonlinear equation with the Atangana-Baleanu- Reimann sense.

$${}^{ABR}_{a}D^{\delta}_{\tau}\varphi(\mu,\tau)-6\varphi\varphi_{\mu}+\varphi_{\mu\mu\mu} = 0, 0<\delta\leq 1,$$

Subject to the initial condition φ(μ,0)=6x.By using the natural transform to both sides of above equation,

$$N\left[{}^{ABR}_{a}D^{\delta}_{t}\varphi(\mu,\tau)-6\varphi(\mu,\tau)\varphi_{\mu}(\mu,\tau)+\varphi_{\mu\mu\mu}(\mu,\tau)=0\right] \quad (49)$$

By using the inverse natural transform to both sides of Eq. (49) and the initial condition,

$$\varphi(\mu,\tau) = -N^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right)N\left[-6\varphi\varphi_{\mu}+\varphi_{\mu\mu\mu}\right]\right] \quad (50)$$

By applying homotopy permutation method on Eq. (50),

$$\sum_{n=0}^{\infty}P^n\varphi_n = -pN^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right)N\left[-6\sum_{n=0}^{\infty}P^nA_n+\sum_{n=0}^{\infty}P^n\varphi_{n\mu\mu\mu}\right]\right], \quad (51)$$

By comparing both sides of the Eq. (51), the following result is obtained,

$$P^0:\varphi_0 = 6x$$

$$p^1:\varphi_1 = -N^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right)N\left[-6p^0A_0+p^0\varphi_{0\mu\mu\mu}\right]\right] \quad (52)$$

$$p^2:\varphi_2 = -N^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right)N\left[-6P^1A_1+p^1\varphi_{1\mu\mu\mu}\right]\right]$$

By the above algorithms,

$$\varphi_0 = 6x$$

$$\varphi_1 = 6^3x\left(1-\delta+\delta\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right) \quad (53)$$

$$\varphi_2 = 6^4x\left(2(1-2\delta+\delta^2)+2(2\delta-2\delta^2)\frac{\tau^{\delta}}{\Gamma(\delta+1)}+2\delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\right),$$

And so on. Therefore, the series solution φ(μ,τ) of nonlinear equation with the Atangana-Baleanu- Reimann sense is given by,

$$\varphi(\mu,\tau) = 6x+6^3x\left(1-\delta+\delta\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right)+6^4x\begin{pmatrix}2(1-2\delta+\delta^2)\\ +2(2\delta-2\delta^2)\frac{\tau^{\delta}}{\Gamma(\delta+1)}+2\delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\end{pmatrix} \quad (54)$$

If we put δ→1 in Eq. (54), we get the approximate and exact solution.

$\varphi(\mu,\tau) = 6x(1 + 6^2\tau + 6^4\tau^2 + \cdots) = \frac{6x}{1-6^2\tau}$ (55)

Example 3 Consider the nonlinear system of time-fractional differential equation in the Atangana-Baleanu Reimann operator:

$$^{ABR}D_\tau^\delta \varphi(\mu,\xi,\tau) - \psi_\mu \vartheta_\xi - 1 = 0, 0 < \delta \leq 1,$$
$$^{ABR}D_\tau^\lambda \psi(\mu,\xi,\tau) - \varphi_\xi \vartheta_\mu - 5 = 0, 0 < \lambda \leq 1,$$
$$^{ABR}D_\tau^\rho \vartheta(\mu,\xi,\tau) - \psi_\xi \varphi_\mu - 5 = 0, 0 < \rho \leq 1,$$

Taking the natural transform on both sides of the initial conditions of Example 3,

$$N\left\{{}^{ABR}_{\ a}D_t^\delta \varphi(\mu,\xi,\tau)\right\} = \left(1 - \delta + \delta\left(\frac{u}{s}\right)^\delta\right) N\{-\psi_\mu \vartheta_\xi - 1\},$$
$$N\left\{{}^{ABR}_{\ a}D_\tau^\lambda \psi(\mu,\xi,\tau)\right\} = \left(1 - \delta + \delta\left(\frac{u}{s}\right)^\lambda\right) N\{-\varphi_\xi \vartheta_\mu - 5\}, \quad (56)$$
$$N\left\{{}^{ABR}_{\ a}D_\tau^\rho \vartheta(\mu,\xi,\tau)\right\} = \left(1 - \lambda + \lambda\left(\frac{u}{s}\right)^\rho\right) N\{-\psi_\xi \varphi_\mu - 5\},$$

Operating with the NT on both sides of (56) gives,

$$\varphi(\mu,\xi,\tau) = -N^{-1}\left(\left(1 - \delta + \delta\left(\frac{u}{s}\right)^\delta\right) N\{-\psi_\mu \vartheta_\xi - 1\}\right),$$

$$\psi(\mu,\xi,\tau) = -N^{-1}\left(\left(1 - \delta + \delta\left(\frac{u}{s}\right)^\lambda\right) N\{-\varphi_\xi \vartheta_\mu - 5\}\right), \quad (57)$$

$$\vartheta(\mu,\xi,\tau) = -N^{-1}\left(\left(1 - \lambda + \lambda\left(\frac{u}{s}\right)^\rho\right) N\{-\varphi_\xi \vartheta_\mu - 5\}\right).$$

By applying homotopy permutation method on Eq. (57),

$$\varphi(\mu,\xi,\tau) = -p\left(N^{-1}\left(\left(1 - \delta + \delta\left(\frac{u}{s}\right)^\delta\right) N\{-\psi_\mu \vartheta_\xi - 1\}\right)\right),$$

$$\psi(\mu,\xi,\tau) = -p\left(N^{-1}\left(\left(1 - \delta + \delta\left(\frac{u}{s}\right)^\lambda\right) N\{-\varphi_\xi \vartheta_\mu - 5\}\right)\right),$$

$$\vartheta(\mu,\xi,\tau) = -p\left(N^{-1}\left(\left(1 - \lambda + \lambda\left(\frac{u}{s}\right)^\rho\right) N\{-\varphi_\xi \vartheta_\mu - 5\}\right)\right) \quad (58)$$

On comparing both sides of the (58),

$$P^0: \varphi_0 = \mu + 2\xi, p^0: \psi_0 = \mu - 2\xi, p^0: \vartheta_0 = -\mu + 2\xi$$

$$p^1: \varphi_1 = -N^{-1}\left(\left(1 - \delta + \delta\left(\frac{u}{s}\right)^{\delta}\right) N\{-\psi_{0\mu}\vartheta_{0\xi} - 1\}\right),$$

$$p^1: \psi_1 = -N^{-1}\left(\left(1 - \delta + \delta\left(\frac{u}{s}\right)^{\lambda}\right) N\{-\varphi_{0\xi}\vartheta_{0\mu} - 5\}\right),$$

$$P^1: \vartheta_1 = -N^{-1}\left(\left(1 - \delta + \delta\left(\frac{u}{\delta}\right)^{\lambda}\right) N\{-\varphi_{0\xi}\vartheta_{0\mu} - 5\}\right),$$

$$p^2: \varphi_2 = -N^{-1}\left(\left(1 - \delta + \delta\left(\frac{u}{s}\right)^{\delta}\right) N\{-\psi_{1\mu}\vartheta_{1\xi} - 1\}\right),$$

$$p^2: \psi_2 = -N^{-1}\left(\left(1 - \delta + \delta\left(\frac{u}{s}\right)^{\lambda}\right) N\{-\varphi_{1\xi}\vartheta_{1\mu} - 5\}\right),$$
$$f^2: \vartheta_2 = -N^{-1}\left(\left(1 - \delta + \delta\left(\frac{u}{s}\right)^{\lambda}\right) N\{-\varphi_{1\xi}\vartheta_{1\mu} - 5\}\right). \tag{59}$$

By the above algorithms,

$$\varphi_0 = \mu + 2\xi, \psi_0 = \mu - 2\xi, \vartheta_0 = -\mu + 2\xi,$$
$$\varphi_1 = 3\left(1 - \delta + \delta\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right),$$
$$\psi_1 = 3\left(1 - \lambda + \lambda\frac{\tau^{\lambda}}{\Gamma(\lambda+1)}\right), \tag{60}$$
$$\vartheta_1 = 3\left(1 - \rho + \rho\frac{\tau^{\rho}}{\Gamma(\rho+1)}\right),$$
$$\varphi_2 = 0, \psi_2 = 0, \vartheta_2 = 0.$$

Therefore, the approximate solution of (60) is given by,

$$\varphi = \mu + 2\xi + 3\left(1 - \delta + \delta\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right),$$

$$\psi = \mu - 2\xi + 3\left(1 - \lambda + \lambda\frac{\tau^{\lambda}}{\Gamma(\lambda+1)}\right),$$

$$\vartheta = -\mu + 2\xi + 3\left(1 - \rho + \rho\frac{\tau^{\rho}}{\Gamma(\rho+1)}\right) \tag{61}$$

If we put $\delta \to 1$ and $\lambda \to 1$ in (61), we reproduce the solution of the problem as follows,

$$\varphi = \mu + 2\xi + 3\tau,$$
$$\psi = \mu - 2\xi + 3\tau, \tag{62}$$
$$\vartheta = -\mu + 2\xi + 3\tau$$

This solution is equivalent to the exact solution in closed form,

$$\varphi = \mu + 2\xi + 3\tau,$$
$$\psi = \mu - 2\xi + 3\tau, \tag{63}$$
$$\vartheta = -\mu + 2\xi + 3\tau$$

5. MAIN RESULTS

5.1. THE TIME-CONTINUOUS TRANSFER FUNCTION

The application of the AB operator (35) or (37) in the elementary FO differential equation (24) yields:

$$T_\alpha\left({}^{AB}_{\alpha}D^{\alpha}_{t}y(t)\right)+y(t)=u(t)$$

Assume homogenous initial condition. Using of (39) in above equation we obtain:

$\frac{T_\alpha b_\alpha s^\alpha}{s^\alpha+a_\alpha}Y(s)+Y(s)=U(s).$ (64)

Consequently, the transfer function $G(s)=\frac{Y(s)}{U(s)}$ takes the following form:

$G_{AB}(s)=\frac{s^\alpha+a_\alpha}{T_{AB}s^\alpha+a_\alpha}.$ (65)

where a_α and b_α are expressed by (42) and (43) respectively, and:

$T_{AB}=1+T_\alpha b_\alpha.$ (66)

The step response of the transfer function is described by the following proposition:

Proposition 1. (The step response of the transfer function using AB operator) Consider the FO transfer function G_AB (s) described by (65).

Its step response takes the following form:

$y_{AB}(t)=1(t)+\left(\frac{1}{T_{AB}}-1\right)E_\alpha\left(-\frac{a_\alpha t^\alpha}{T_{AB}}\right)$ (67)

where T_AB is expressed by (66).

Proof. The transfer function (65) can be expressed as the sum of two following transfer functions:

$$G_{AB1}(s)\quad \frac{s^\alpha}{T_{AB}s^\alpha+1},$$
$$G_{AB2}(s)\quad \frac{a_\alpha}{T_{AB}s^\alpha+1}. \qquad (68)$$

The step response we are looking for is the sum of step responses of both components (68). Denote these responses as y_AB1 (t) and y_AB2 (t), respectively. They are equal:

$y_{AB1,2}(t)=L^{-1}\left\{\frac{1}{s}G_{AB1,2}(s)\right\}$ (69)

The step response y_AB1 (t) is obtained using Equation (1.34), page 11 in book [17]. It takes the following form:

$y_{AB1}(t)=\frac{1}{T_{AB}}E_\alpha\left(\frac{-a_\alpha t^\alpha}{T_{AB}}\right).$ (70)

Next, the step response y_AB2 (t) we obtain using (26):

$y_{AB2}(t)=1(t)-E_\alpha\left(\frac{-a_\alpha t^\alpha}{T_{AB}}\right).$ (71)

After adding (70) to (71) we obtain (67) and the proof is completed.

Next the steady-state response of the considered transfer function (65) is described by the following remark.

Remark 1. (The steady-state response of the time-continuous transfer function)

Consider the transfer function using AB operator (65). Its steady-state response is equal:

$$y_{ss}=1.$$

Proof. The Laplace transform of the step response of the transfer function (66) is as follows:

$Y(s)=\frac{1}{s}G_{AB}(s).$ (72)

The steady-state value of (72) is obtained using Final Value Theorem (FVT):

$y_{ss}=\lim_{s\to 0}sY(s)=\lim_{s\to 0}G_{AB}(s)=\lim_{s\to 0}\frac{s^\alpha+a_\alpha}{T_\alpha b_\alpha s^\alpha+a_\alpha}$ (73)

An interesting issue is to compare the proposed transfer function (65) with the well-known transfer function using Caputo operator (25). To do this the following "quasi-norms" H describing the distance between step responses of both transfer functions are proposed:

$$H_{max} = \max_{0\geq t\geq T_f} |y_C(t) - y_{AB}(t)| \quad (74)$$

$$H_2 = \int_0^{T_f} \left(y_C(t) - y_{AB}(t)\right)^2 dt \quad (75)$$

where T_f is the final time of the response calculation's fundamental parameter describing dynamics of each system is damping rate ξ that can be easily defined for scalar systems considered here. It is equal for each considered transfer function:

$$\xi_C = \frac{1}{T_\alpha} \quad (76)$$

$$\xi_{AB} = \frac{a_\alpha}{T_{AB}} \quad (77)$$

5.2. THE APPROXIMATED, DISCRETE TRANSFER FUNCTION

The discrete-time transfer function using FOBD operator with fixed memory length L is obtained by employing of (32) in (65):

$$G_{ABL}(z^{-1}) = \frac{N_L(z^{-1})}{D_L(z^{-1})}, \quad (78)$$

where:

$$N_L(z^{-1}) = h^{-\alpha} \sum_{l=0}^{L} d_l\, z^{-l} + a_\alpha$$
$$D_L(z^{-1}) = h^{-\alpha} T_{AB} \sum_{l=0}^{L} d_l\, z^{-l} + a_\alpha \quad (79)$$

For each memory length it is expressed as:

$$G_{AB\infty}(z^{-1}) = \frac{N_\infty(z^{-1})}{D_\infty(z^{-1})}, \quad (80)$$

where:

$$N_\infty(z^{-1}) = h^{-\alpha} \sum_{l=0}^{\infty} d_l\, z^{-l} + a_\alpha,$$
$$D_\infty(z^{-1}) = h^{-\alpha} T_{AB} \sum_{l=0}^{\infty} d_l\, z^{-l} + a_\alpha. \quad (81)$$

The Z transform of the step response of both considered transfer functions takes the following form:

$$Y_{L,\infty}(z^{-1}) = \frac{G_{FOBD}^{+(z^{-1})}}{1-z^{-1}} \quad (82)$$

and consequently, the step response of the discrete, approximated transfer function is as follows:

$$y_{L,\infty}(k) = Z^{-1}\{Y_{L,\infty}(z^{-1})\}, \quad (83)$$

where k=1,2,… denotes discrete time instants. The formula (83) can be solved numerically using e.g. MATLAB, which was used for numerical validation of results discussed in the next section.

5.3. THE CONVERGENCE OF THE DISCRETE APPROXIMATION

The Rate of Convergence (ROC) of the proposed, discrete, approximated model can be defined as follows:

Definition 16. (The Rate of Convergence)

ROC of the discrete transfer function (78) constructed with the fixed memory length L is equal to its steady state value in the discrete transfer function:

$$ROC_L = y_{ssL} \quad (84)$$

where $y_{ssL} = \frac{S_L + h^\alpha a_\alpha}{T_{AB} S_L + h^\alpha a_\alpha}$.

It is obvious that $\lim_{L\to\infty} ROC_L=1$. The ROC is a function of parameters of FOBD: sample time h and memory length L. It is also a function of parameters of the model: fractional order α and time constant T_α.

The value of the sample time h assuring the minimum value Δ_L of ROC is described by the following proposition:

Proposition 3. (The value of sample time h assuring the minimum value of ROC)

Consider the discrete FO transfer function (78). The minimum value of the sample time h assuring the minimum, predefined value of Δ_L is described as follows:

$$h \geq \left(\frac{S_L(1-\Delta_L T_{AB})}{a_\alpha(\Delta_L - 1)}\right)^{\frac{1}{\alpha}}$$

Proof. The minimum predefined value of ROC using y_{ssL} is expressed as follows:

$$\begin{aligned} &\Delta_L \geq \frac{S_L + h^\alpha a_\alpha}{T_{AB} S_L + h^\alpha a_\alpha} \Leftrightarrow \\ &\Leftrightarrow h^\alpha a_\alpha(\Delta_L - 1) \geq S_L(1-\Delta_L T_{AB}) \Leftrightarrow \\ &\Leftrightarrow h^\alpha \geq \left(\frac{S_L(1-\Delta_L T_{AB})}{a_\alpha(\Delta_L - 1)}\right) \Leftrightarrow \\ &\Leftrightarrow h \geq \left(\frac{S_L(1-\Delta_L T_{AB})}{a_\alpha(\Delta_L - 1)}\right)^{\frac{1}{\alpha}}. \end{aligned} \tag{85}$$

6. SIMULATIONS

6.1. THE TIME CONTINUOUS TRANSFER FUNCTION

Firstly, the step responses using analytical formula (67) were examined. Time trends obtained using MATLAB for different values of fractional order α and time constants T_α are shown in Figures 1. These analytical responses will be used as a reference to estimate the quality of the discrete approximation.

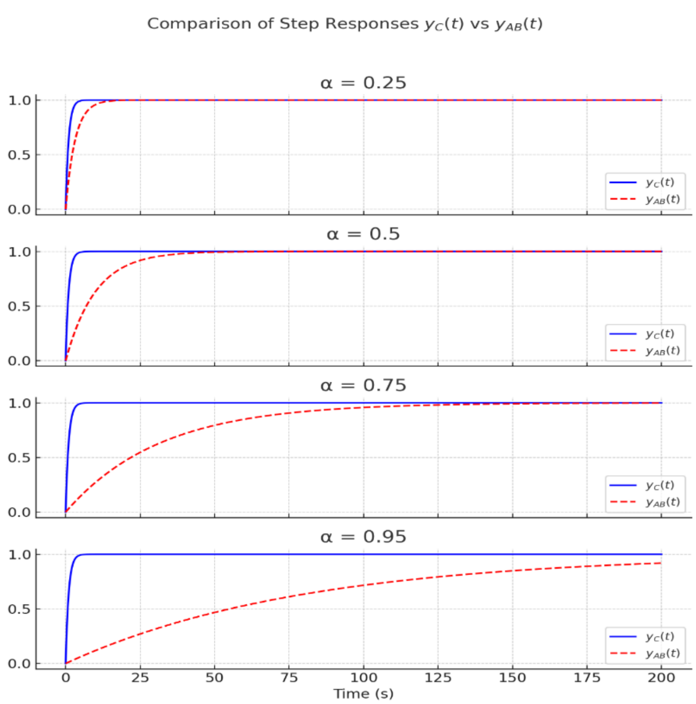

Figure 1 The comparing of step responses $y_C(t)$ vs $y_{AB}(t)$ for T_α=1s,T_f="100" s and α="0.25" , 0.50, 0.75, 0.95 (top-bottom)

Table 1: Quasi norms (74) and (75) for T_α=1s,T_f="100" s and various fractional orders α

α	H_{max}	H_2
0.25	0.4780	1.1417
0.50	0.3900	0.1888
0.75	0.2248	0.0167
0.95	0.0490	$7.9049e-04$

These quasi-norms reflect how system behaviour can differ under varying fractional orders α, with (74) growing exponentially and (75) decreasing rapidly for small α.

6.2. THE APPROXIMATED TRANSFER FUNCTION USING FOBD

In this section the discrete transfer function using FOBD approximation was examined. Its accuracy was estimated using known Integral Absolute Error (IAE) cost function:

$$IAE = h\sum_{k=1}^{K}|y_{AB}(kh) - y_L(kh)|, \qquad (86)$$

where y_AB (kh) is the analytical response (67) and y_L (kh) is the step response of approximation (83). For fixed α and T this cost function is a function of memory length L and sample time h. Its 3D plot for L="100"-"500" and h="0.1"-"10" s is shown in Fig. 2.

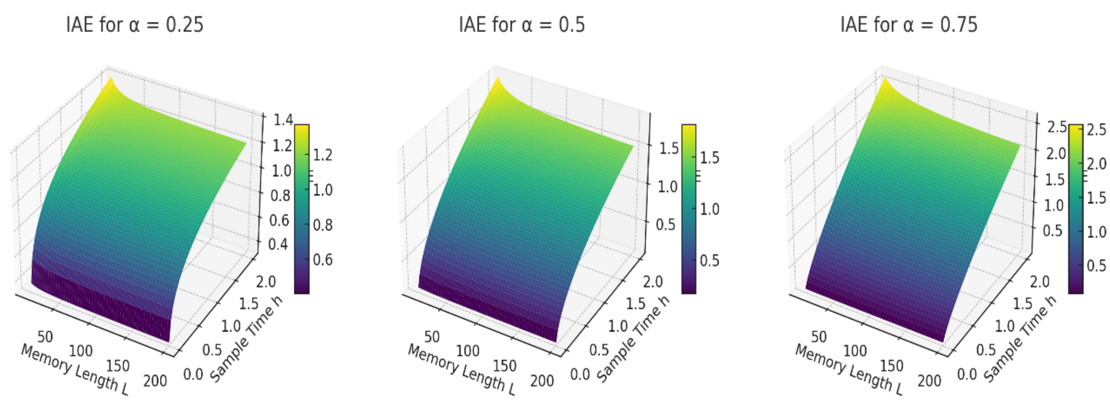

Figure 2 The IAE cost function as a function of memory length L and sample time h for α="0.25" , "0.50","0.75"

Comparison of y_AB (t) and y_L (kh) for different values of fractional order α is shown in Fig. 2 and respective ISE values are given in Table 2.

Table 2: The cost function (89) for T_α=1s,h=1s,L="100" and various fractional orders α

α	0.25	0.50	0.75
ISE	0.1372	0.3312	0.4554

7. CONCLUSION AND DISCUSSION

In this article, the natural homotopy permutation method was presented and the following results were obtained:

- The method is effective and efficient in solving fractional differential equations with the Atangana-Blaenau Reimann operator.
- The approximate solutions obtained by this method approximates the exact solution when δ,λ=1.
- The method can solve linear and nonlinear equations.

The most significant observation regarding the time continuous transfer function (65) is that, for fractional order α near 1.0, its step response tends to resemble the step response of the "classic" FO transfer function utilizing the C operator (25). Table 1 and Fig. 1 demonstrate this. Based on this finding, it may be inferred that the suggested transfer function is only useful for fractional orders α that are substantially smaller than 1.0. The "classic" transfer function (25) is easier to build and guarantees fairly similar performance in terms of the step response for fractional order α near to 1.0.

The suggested model has a significant advantage, as demonstrated by the discrete transfer function analysis using the FOBD approximation (78, 80). Specifically, short memory length L is linked to the good accuracy it achieves for lengthy sample times h. In the 3D plots 1 and 2, it is visible. When it comes to digital implementation on a limited platform, such as a PLC or microcontroller, this trait can be quite helpful.

In order to achieve the same accuracy and convergence, the sample time h must be shortened while the memory length L must be increased.

CONFLICT OF INTERESTS

None.

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