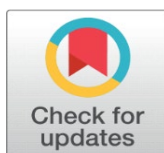
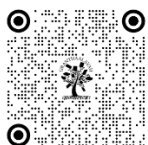


# EXPLORING CHIP-FIRING GAMES ON EULERIAN AND HAMILTONIAN GRAPHS APPROACH

Kavita Hiremath<sup>1</sup>✉, Namrata Kaushal<sup>2</sup>✉, Jyoti Gupta<sup>2</sup>✉

<sup>1</sup>Department of Mathematics, Mansarovar Global University, Billkisganj, Sehore, Madhya Pradesh, India 466111

<sup>2</sup>Department of Mathematics, Indore Institute of Science and Technology, Indore, Madhya Pradesh, India 453331



## Corresponding Author

Kavita Hiremath,  
[kavita.hiremath1@gmail.com](mailto:kavita.hiremath1@gmail.com)

## DOI

[10.29121/shodhkosh.v5.i1.2024.5931](https://doi.org/10.29121/shodhkosh.v5.i1.2024.5931)

**Funding:** This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

**Copyright:** © 2024 The Author(s).

This work is licensed under a [Creative Commons Attribution 4.0 International License](https://creativecommons.org/licenses/by/4.0/).

With the license CC-BY, authors retain the copyright, allowing anyone to download, reuse, re-print, modify, distribute, and/or copy their contribution. The work must be properly attributed to its author.



## ABSTRACT

Based on the graph structure and chip distribution on its vertices, three distinct types of dynamic models emerge. Among these, Chip-Firing Games have garnered significant attention due to their wide applicability across various mathematical fields, including algebra, combinatorics, dynamical systems, statistics, algorithms, and computational complexity. In this paper, we explore the characterization properties of configuration spaces, which are organized by predecessor relations. We introduce Chip-Firing Games (CFGs) as a method, termed the "Probability Abacus," to compute absorbing probabilities through vector addition techniques. We demonstrate that the termination state of any CFG within an absorbing Markov chain, characterized by a rational transition matrix, which is independent of the firing order and critical loading reappearing at the termination state.

**Keywords:** Graph, Lattice, ULD Lattice, Walk, Markov Chain

## 1. INTRODUCTION

The chip-firing game introduced by Bjorner and Lovasz [1], is one of the classical discrete dynamic models which defined on directed graphs, or support graphs. Later this game was generalized to undirected graphs by Bjorner, Lovasz, and Shor [2]. Magnien et al. [5] demonstrated that every distributive lattice can be represented as a chip-firing game and that all generalized chip-firing games can be classified as uniform lattice distributive (ULD). Kimmo Eriksson [9] showed that every chip-firing game is strongly convergent, meaning it either continues indefinitely or reaches a unique termination state where no further firing is possible. This paper focuses on chip-firing games that eventually reach a unique fixed point.

In this paper we consider only those CFGs that reach up to the unique fixed point. Random walks on graphs are approximated by computing the expected hitting time, or probable number of random moves required to go from one vertex to another. Although random walks are valuable in mathematics and computer science, probabilistic methods

may lack precision for certain applications. Recent advancements in deterministic simulation methods offer more accurate ways to compute the position of an object at any given stage [13, 10].

Engel [3,4] introduced the notion of "probabilistic abacus," a chip-firing procedure designed to determine absorption probabilities and access times of certain Markov chains through combinatorial techniques. In this context, we focus on interpretation of chip firing game (CFGs) within vector-addition languages. The results from these vector-addition languages are equivalent to those from ULD lattices, leading generalized chip-firing games for these language. Vector addition languages were introduced by Karp and Miller [15]. They are also known as general Petri nets [16] and are one of the most popular formal methods for analysis and representation of parallel processes [17]. We will only use them for splitting absorbing probability.

In recent research, C. Merino [7] highlighted numerous connections of the chip-firing game with various aspects of combinatorics and theoretical physics, including its relationships with the Tutte polynomial, group theory, greedoids with repetition, and matroids. M. Baker and F. Shokrieh [12] explored the interface between chip-firing games and potential theory on graphs, characterizing reduced divisors as solutions to an energy minimization problem and presenting an efficient algorithm for computing them. Z. Scully et al. [18] focused on firing sequences in the parallel chip-firing game, providing a comprehensive characterization of periodic firing sequences and introducing the concept of motors to explore local game behavior. Additionally, A. Kelley in 2016 extended the chip-firing game to a two-dimensional line with distinct chips, establishing that final configurations can exhibit either sorted or unsorted arrangements for odd numbers of chips and conjecturing that they tend to be sorted for even numbers of chips (see [5]). H. Zhong [8] delved into fundamental properties of the game using the graph Laplacian, examining different firing mechanisms and sink effects, and investigating combinatorial categorizations of chip configurations and the impact of value changes on game stability and continuation.

In this paper we define CFG on absorbing Markov chain with rational transition probabilities ordered by predecessor relations with three absorbing states. We will use results of [15] and show that termination state of game does not depend on order of firing to vertex. Furthermore, with the same initial configuration a game played with two different strategies game reaches to critical loading which will be in the form of vector addition language.

## 2. PRELIMINARIES AND TERMINOLOGIES

### 2.1. LATTICE

Let  $L = (X, \leq)$  be a finite partial order set,  $L$  be a lattice if any two elements  $x, y \in X$  of  $L$  has a least upper bound (join), denoted as  $x \vee y$  and a greatest lower bound (meet) denoted as  $x \wedge y$ . If there is precisely one upper cover for  $x$  in  $X$ , then  $x$  is a meet-irreducible. If there is exactly one lower cover for the element  $x$ , then  $x$  is a join-irreducible. Let  $M$  and  $J$  denote the collections of the meet-irreducible and the join-irreducible of  $L$ , respectively. Let  $M_X = \{m \in M: x \leq m\}$  and  $J_X = \{j \in J: j \leq x\}$ .

Proposition 2.1 [6]: Let  $L$  be a lattice. Any element  $x$  of  $L$  is the join of the join-irreducible that are smaller than itself, and the meet of the meet-irreducible that are greater than itself:

$$x = \bigvee \{j \in J: j \leq x\} = \bigwedge \{m \in M, x \leq m\} \quad \text{i.e. } x = \bigvee J_X = \bigwedge M_X$$

### 2.2. ULD LATTICE

Let  $L = (X, \leq)$  be a poset  $L$  is an upper locally distributive lattice (ULD) if  $L$  is a lattice and each element has a unique minimal representation as meet of meet-irreducible, i.e., there is a mapping  $M: L \rightarrow L = \{m \in L: m \text{ is meet-irreducible}\}$  with the properties:

$$* x = \bigwedge M_X((\text{representation}) * x = \bigwedge A \Rightarrow M_X \subseteq A(\text{minimal}))$$

If  $L$  be a ULD lattice with  $M$  be a set of meet-irreducible then consider the map  $\uparrow x M := \uparrow x \cap M$ . The definition of meet-irreducible implies that  $x = \bigwedge \uparrow x M$  for all  $x$ , where  $\uparrow x$  is meet-irreducible above  $x$ .

### 2.3. RANDOM WALKS

A walk in a graph or digraph is a sequence of vertices  $\{v_1, v_2, v_3, \dots, v_k\}$ , not necessarily distinct. Now, if we place some objects corresponding to each stage on each vertex and edge shoes probability of moving objects from one vertex

to other given by Markova transition matrix  $m_{ij}$ , at each stage occurs a sequence of adjacent vertices. This sequence represents the position of the object at a given stage, which is called random walk.

## 2.4. ABSORBING MARKOV CHAIN

A state  $s_i$  of a Markov chain is called absorbing if it is impossible to leave it (i.e.,  $p_{ij}=1$ ). A Markov chain is absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state.

In this paper we define chip-firing game as a process called Engel [7,8] “probabilistic abacus” on supported graph, then we will prove some results of absorbing probability by properties of ULD lattice generated by configuration space with predecessor relation followed by firing sequence. For which we create one node for each state and put some chips at the nodes corresponding to the non-absorbing called transitions states. Transitions probability of moving chip from one vertex to another is  $p_{ij} = r_{ij}/r_i, \forall j$ , where

$Sr_i, r_{i1}, r_{i2}, \dots, r_{in}$  are integers. If there were  $r_i$  chips at node  $i$  we could ‘fire’ or ‘make a move’ in node  $i$ . To begin the game we require initial configuration in probability abacus, called critical loading.

## 2.5. CRITICAL LOADING

Critical loading is one in which each node has one less chip that it needs to fire, i.e.  $c_i = r_i - 1$ .

## 2.6. VECTOR ADDITION LANGUAGE

A vector-addition language is a language  $L(M, \mu)$  given by an alphabet  $M \subset \mathbb{R}^d$  and its starting configuration is  $\mu \in \mathbb{R}^d \geq 0$ . A word  $s = (x_1, x_2, x_3, \dots, x_k)$  is in  $L(M, \mu)$  if  $x_i \in M$  and  $\mu + x_1 + \dots + x_k \geq 0$  for all  $1 \leq i \leq k$ .

## 3. RESULTS USED

For terminology, notations and properties of trefer [12] and for absorbing Markova chain properties refer [16].

**Theorem 3.1[6]:** A lattice is distributive if and only if it is isomorphic to the lattice of the ideals of the order induced by its meet-irreducible.

**Theorem 3.2[6]:** The lattice of the configuration space of a CFG is ULD.

**Lemma 3.3 [6]:** Let  $L$  be a ULD and  $x, y \in L$ . We have  $x < y$  if and only if  $|\uparrow x_M \setminus \uparrow y_M| = 1$ .

## 4. MAIN RESULTS

**Theorem 4.1:** If defined CFG on absorbing Markov chain with rational transition probabilities ordered by predecessor relations and its configuration space forms ULD lattice then termination state of CFG does not depend on the order of firing in which moves are made.

**Proof:** Suppose, if possible, termination state depends on the order of moves- Suppose game be played by two strategy and we get  $C(L_1) = \{\mu_0, \mu_1, \mu_2, \dots, \mu_m, \leq(p)\}$  configuration space with predecessors relations  $\leq p$  from first strategy &  $C(L_2) = \{\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_n, \leq(p)\}$  configuration space from strategy second. And their respective shot sets ordered by inclusion order will be  $S(L_1) = \{v_0, v_1, v_3, \dots, v_m, <\}$  and  $S(L_2) = \{u_0, u_1, u_2, \dots, u_n, <\}$ . Each shot set will be isomorphic to their corresponding configuration space i.e.  $C(L_1) \cong S(L_1)$  and  $C(L_2) \cong S(L_2)$ . Both configuration spaces are ULD lattice. Now we compare both strategies-let for  $x, y \in S_1$ .

Let  $x < y$  i.e.  $x$  covers  $y$  since set of configurations are ULD with predecessors relations and shot set be isomorphic to respective configuration space. Hence by lemma [2.2]  $|\uparrow x_M \setminus \uparrow y_M| = 1$ , because lattice are ranked so number of meet irreducible for respective elements will be same. Now, let by strategy two we get shot set  $S_2$  for which  $x, y \in S_2$  such that  $x \not< y$  then by lemma [2.2]  $|\uparrow x_M \setminus \uparrow y_M| \neq 1$ . Let us suppose  $|\uparrow x_M \setminus \uparrow y_M| \geq 2$ .  $\wedge m_x < \wedge \uparrow y_M$ . There has to be some  $m_1 \in M_x / (\uparrow y_M)$ .

Let  $z = \wedge(\uparrow x_M - m_1)$ . Since in ULD lattice for every element there is unique inclusion-minimal set  $M_x \subseteq M(S_1)$  such that  $x = \wedge M_x$ . So we have  $z = \wedge(\uparrow x_M - m_1) > x$ . Since  $(\uparrow x_M - m_1) \supseteq \uparrow y_M$  therefore we have  $z \leq y$ . Now let  $m_2$  be an element which differs from  $m_1$  and belongs to  $\uparrow x_M \setminus \uparrow y_M$  Follows  $m_2 \in \uparrow z_M$  and  $m_2 \notin \uparrow y_M$  hence  $z \neq y$ . This

implies that  $x < z < y$  i.e. pair  $x, y$  is not in covering relation, which shows contradiction to strategy one which follows lemma [2.5.3]. Hence we conclude that our assumption that termination state depends on order of firing is wrong, and we can say it does not depend on order of firing.

**Theorem 4.2:** Let  $G$  be any digraph, let  $\mu_0, \mu_1, \mu_2, \dots, \mu_m$  be a sequence of chip configurations on  $G$ , ordered each of which is a predecessors of the one before, and let  $\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_n$  be another such sequence with  $\mu_0 = \varphi_0$

- 1) If configuration space with predecessors relations  $\leq(p)$  is a ULD then, and  $\mu_m$  is a terminating configuration then  $n \leq m$  and moreover no firing sequence may have more than  $m$  meet-irreducible.
- 2) If  $\mu_m$  and  $\varphi_n$  both terminating configuration then  $n = m, \mu_m = \varphi_n$  and in each firing sequences game terminates when critical loading reoccurs.

Proof: Part 2 is an immediate corollary of part 1; let part 1 fail so we prove our claim by taking an opposite assumption with  $m+n$  minimal. Suppose in strategy one the vertex  $[v]_i$  is fired when configuration  $\mu_{i-1}$  becomes  $\mu_i$  (because  $\leq(p)$  is a predecessor relation) moreover in strategy two the vertex  $[u]_j$  will be fired when configuration  $\varphi_{j-1}$  becomes  $\varphi_j$ . In strategy two the vertex  $[u]_1$  must be fired at some stage in the sequence of its configuration. Since  $\mu_m$  is the configuration in strategy one at which game is terminated, then  $[v]_i$  must be equal to  $u_1$  then  $[v]_i, [v]_1, [v]_2, \dots, [v]_{i-1}, [v]_{i+1}, [v]_m$  be the valid firing sequence which turns configuration  $\mu_0$  into  $\mu_m$  from our first main result with the same number of firing in different order game can be terminated at same configuration i.e.  $\mu_m$ . So we can see that the firing sequence  $[v]_1, [v]_2, \dots, [v]_{i-1}, [v]_{i+1}, [v]_m$  and  $u_2, u_3, \dots, u_n$  will be contradicting to the minimality of lemma starting with same initial configurations, which proves part 2.

### For better explanation of our results, we take one example

Let us define a chip-firing game on absorbing Markov chain with rational transition matrix. Corresponding to given transition matrix, we create some nodes  $\{1, 2, 3\}$  which are the transition state (vertex) from where chip can move on other adjusting vertices according to given firing rule (transition probability) and each transition node has one less chip that it needs to fire i.e.  $c_i = r_i - 1, \{4, 5\}$  which are absorbing states from where once a chip enters in this state it is impossible to leave this state, and  $\{0\}$  is firing node which contains large number of chips follow firing rule that node (vertex) 0 may fire only if no other node (vertex) can fire.

As per defined Chip-firing game critical loading by placing chips at transition nodes  $\{1, 4\}$  will be  $(9, 0, 0, 9, 0)$ :

Now we play defined game by two different strategies, and will compare result

#### Strategy 1

We start with critical loading  $(9, 0, 0, 9, 0)$  then fire node (vertex) 0 and then node (vertex) 1.

$$\begin{aligned} (9, 0, 0, 9, 0) &\xrightarrow{0} (10, 0, 0, 9, 0) \xrightarrow{1} (4, 2, 3, 9, 1) \xrightarrow{0} (4, 2, 3, 10, 1) \xrightarrow{4} (5, 7, 4, 1, 3) \xrightarrow{9 \text{ times}} (5, 7, 4, 10, 3) \xrightarrow{4} (6, 12, 5, 1, 5) \\ &\xrightarrow{9 \text{ times}} (6, 12, 5, 10, 5) \xrightarrow{4} (7, 17, 6, 1, 7) \xrightarrow{9 \text{ times}} (7, 17, 6, 10, 7) \xrightarrow{4} (8, 22, 7, 1, 9) \xrightarrow{9 \text{ times}} (8, 22, 7, 10, 9) \\ &\xrightarrow{4} (9, 27, 8, 1, 11) \xrightarrow{8 \text{ times}} (9, 27, 8, 9, 11) \end{aligned}$$

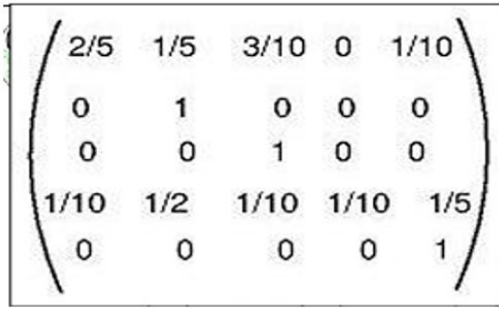
#### Strategy 2

We start with critical loading  $(9, 0, 0, 9, 0)$  then fire node (vertex) 0 and then node (vertex) 4.

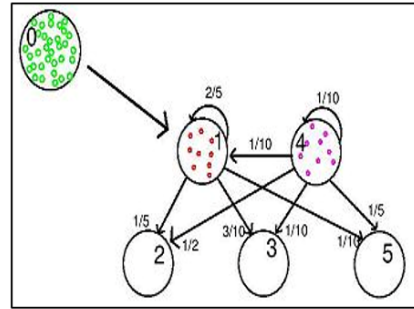
$$\begin{aligned} (9, 0, 0, 9, 0) &\xrightarrow{0} (9, 0, 0, 10, 0) \xrightarrow{4} (10, 5, 1, 1, 2) \xrightarrow{1} (4, 7, 4, 1, 3) \xrightarrow{9 \text{ times}} (4, 7, 4, 10, 3) \xrightarrow{4} (5, 12, 5, 1, 5) \\ &\xrightarrow{9 \text{ times}} (5, 12, 5, 10, 5) \xrightarrow{4} (6, 17, 6, 1, 7) \xrightarrow{9 \text{ times}} (6, 17, 6, 10, 7) \xrightarrow{4} (7, 22, 7, 1, 9) \xrightarrow{9 \text{ times}} (7, 22, 7, 10, 9) \\ &\xrightarrow{4} (8, 27, 8, 1, 11) \xrightarrow{8 \text{ times}} (8, 27, 8, 9, 11) \xrightarrow{0} (9, 27, 8, 9, 11) \end{aligned}$$

From both firing sequence it is clear that after firing node (vertex) 1 one time, node (vertex) 4 five times and node zero 46 times we stop, because transition node  $\{1, 4\}$  have exactly the same loading as at the start. Which can be read

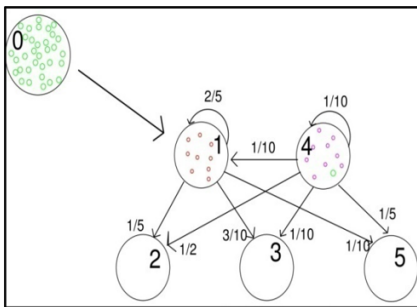
with the help of vector addition language i.e.  $(9,0,0,9,0)+(0,27,8,0,11)$  from final configuration at which game terminates critical loading reoccurs. Thus absorbing probabilities will be  $p_{12} = \frac{27}{46}$ ,  $p_{13} = \frac{8}{46}$ ,  $p_{15} = \frac{11}{46}$



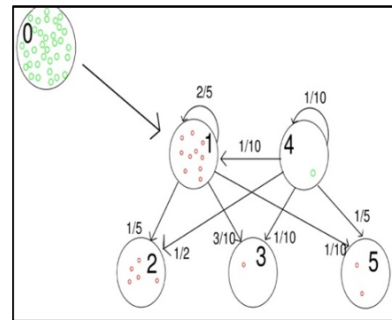
**Figure 2.2** Rational transition probability matrix (2,3,5) are absorbing state



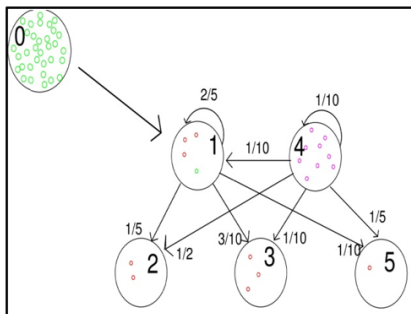
**Figure 2.3** 0 is firing node, (1,4) are transition state and (2,3,5) are absorbing state



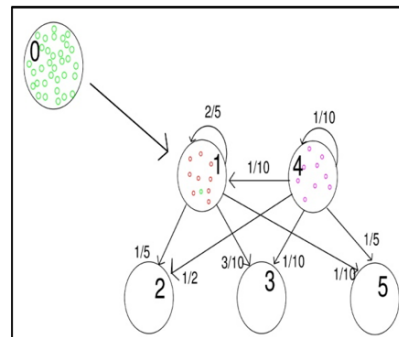
**Figure 2.4** Firing 1



**Figure 2.5** Firing from 0 to 1



**Figure 2.6** Firing from 0 to 4



**Figure 2.7** Firing 4

From the above examples it is clear that if a chip-firing game is played with same initial configuration by different strategy then CFG reaches to a fixed point called final configuration and the state is called termination state. Also, in each firing sequences each vertex fires the same number of times to terminate the game in both strategies.

## 5. CONCLUSION AND FUTURE SCOPE

From the above examples it is clear that if a chip-firing game is played with same initial configuration by different strategy then CFG reaches to a fixed point called final configuration and the state is called termination state. Also, in each firing sequences each vertex fires the same number of times to terminate the game in both strategies. Since during the firing sequence configuration follows the predecessor relation which is reflexive, transitive and closed under predecessor relation and hence constructs lattice. In this lattice all the finite chains among fixed end points have same

length which shows that lattice is ranked hence configuration space with predecessor relation called ULD lattice. In this paper we have proved some results which are based on absorbing probability by using properties of ULD lattice.

The paper appears to delve into the application of Chip-Firing Games (CFGs) to various mathematical fields and explores the characterization of configuration spaces using a method called the 'Probability Abacus'. Future research on Chip-Firing Games (CFGs) could explore new applications in various mathematical fields and algorithm design, as well as improve methods for computing absorbing probabilities. Expanding CFGs to more complex models and applying them to real-world systems could offer fresh insights. Additionally, developing educational materials and fostering collaborations can enhance the understanding and practical use of CFGs across different areas.

## CONFLICT OF INTERESTS

None.

## ACKNOWLEDGMENTS

We are extremely thankful to all those who played a role in the success of this research work. We would like to express our gratitude towards the management for the encouragement and all the supporting staff for their invaluable input and support throughout the research process.

## REFERENCES

- A.Bjorner, L.Lovasz, "Chip-firing games on directed graphs". J. Algebraic Combinatory. 1, (1992), 304-328.
- A.Bjorner, L.Lovasz, W. Shor, "Chip-firing games on graphs" E.J. Combinatorics.12 ,(1991), 283-291.
- A.Engel, "The probabilistic abacus" Educ.Stud. In Math. 6 (1975), 1-22.
- A.Engel, "Why does the probabilistic abacus work" Educ. Stud. In Math. 7, (1976), 59-69.
- A. Kelley, "Chip Firing Games", April 24, (2016).
- C. Magnien, H.D.Phan, L.Vuillon, "Characterization of lattices induced by (extended) chip-firing Games". Discrete Mathematics. Theory of Computer Science, (2001), 229-244.
- Criel Merino, "The chip-firing game", Discrete Mathematics 302 (2005), 188 – 210.
- H. Zhong, "Introduction to the chip-firing game", August 29, (2022).
- J.Esparza, M.Nielsen, "Decidability issues for petrinets" a survey, Bulletin of the European Association for Theoretical Computer Science 52 (1994), 245-262.
- J.Propp, "A whirling tour of chip-firing and rotor-routing", DIMACS Workshop on Puzzling Mathematics and Mathematical Puzzles (2007).
- Kimmo Eriksson, "Strongly Convergent Games and Coxeter Groups". PhD thesis, Kungl Tekniska Hogskolan, Sweden, (1993).
- M. Baker, F. Shokrieh, "Chip-firing games, potential theory on graphs, and spanning trees", Journal of Combinatorial Theory, Series A, 120 (1), (2013), 164-182
- M.Kleber, "Goldbug variations" Mathematical Intelligencer, 27 (1), (2005), 1-20
- M.Latapy, H.D. Phan, "The lattice structure of chipfiring games". Physica D, 115, (2001), 69-82.
- R.M.Karp, R. E.Miller, "Parallel program schemata", J. Computer System Science, 3, (1969), 147-195.
- W.Reisig, "Petrinets", EATCS Monographs on Theoretical Computer Science, 4, (1985).
- W.Feller, "An Introduction to Probability Theory", 1, (1957).
- Ziv Scully, Tian-Yi Jiang, Yan X Zhang, "Firing Patterns in the Parallel Chip-Firing Game", Discrete Mathematics and Theoretical Computer Science proc., (2014), 537-548.