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A STUDY ON SET ALGEBRAIC OPERATIONS CHARACTERISTICS WITH INTUITIONISTIC FUZZY SUBGROUPS UNDER THE CONDITIONS OF INTUITIONISTIC FUZZY TOPOLOGICAL VECTOR SPACE

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ABSTRACT

In this work, a study on intuitionistic fuzzy topology and algebra behavior of IFSGs in context of intuitionistic fuzzy topological vector spaces (IFTVS) is presented. Especially the case of normal space is discussed in terms of the union and intersection algebraic operations of level intuitionistic subgroups. The study, thus presents a special case of union and intersection among IFSGs, examining their structural constitution and impact within the larger network of IFTVS. The findings show that the nice algebraic and topological properties persist under the intersection of any level IFSGs, producing consistently well-defined algebraic and topological characteristics. The union operation does not naturally preserve subgroup structure unless certain containment conditions are satisfied, highlighting the robust dichotomy between these two operations. Theoretical justifications are tutored to elucidate these findings, underlining their efforts been made to towards future real-time mathematical and computational applications. The scope is only limited to conceptual development and formal analysis. The approaches presented and resulting conclusions are instrumental for practical model-building in fuzzy algebra. Generalized containment conditions, graded membership thresholds, fuzzy closure operators, and more could be used to further investigate IFSG families and help develop a general theory to elucidate their behavior. Moreover, the generalization of this framework to more generalized mathematical structures like IFβ-normal spaces and IF-modules can increase the practical relevance of these results through the multiple fields like topological group theory and applied fuzzy systems.

Keywords: Union Level Subgroups, Intersection Level Subgroup, Intuitionistic Level Subgroup Operations, Normal Topological Intuitionistic Spaces

1. INTRODUCTION

Fuzzy sets, based on the fundamental set hypothesis, are the set of elements that allows them to be partially included within a set. This theory, introduced by Zadeh in 1965, is widely used in various fields such as medical diagnosis, gas pipeline vulnerability assessment, trip time analysis, and neural network modelling. Fuzzy sets are more effective in addressing uncertainty and vagueness in physical problems compared to classical fuzzy set theory (Shuaib et al., 2020).

A. Rosenfeld's pioneering register of 1971 expanded Zadeh's fuzzy set theory into the area of abstract algebra by introducing the idea of "fuzzy group". His vision was to extend a vague subset of a specific group by giving a degree of membership (between 0 and 1) to each element of the group, so that the group actions are "maintained" in a fuzzy manner (Soni et al., 2024).

The intuitionistic fuzzy subgroup is particularly useful in real-world transportation issues, as it can model the configuration of control at the intersection of two one-way roadways. The definition of subgroup is expanded through intuitionistic fuzzy subgroups, allowing membership and non-membership degrees for elements, providing improved precision for analyzing uncertain subgroup-like structures without absolute accuracy (Abbas, 2021).

Rosenfeld and Das redefined fuzzy subgroups within fuzzy sets in 1971 and 1981 respectively. Mukherjee and Bhattacharyya in 1984 investigated the normality of a fuzzy subgroup and created fuzzy cosets. Fuzzy logic has been applied in various areas, including logic programming, decision-making, financial services, psychological assessments, medical diagnosis, career guidance, and artificial intelligence (Yuan et al., 2010).

A major problem in intuitionistic fuzzy sets is the establishment of appropriate topological spaces. Coker tackled this with Chang's fuzzy topology. Later, other researchers expanded these ideas and created (r, s)-connected fuzzy sets in an attempt to analyze connectedness under threshold levels. They further extended to intuitionistic fuzzy points and fuzzy neighborhoods, facilitating the study of various generalized forms of fuzzy connectedness in Coker's structure (Abdy et al., 2021).

Additionally, establishment of a suitable definition of intuitionistic fuzzy normed space is also challenging while working on this problem area. This issue has been examined by Saadati and Park. They have proposed and examined the concept of intuitionistic fuzzy normed spaces (Saadati & Park, 2006).

Anticipating a deep understanding on the importance and suitable purpose that the intuitionistic fuzzy topological vector spaces (the study aims to explore normed spaces too) as serve in the applications of intuitionistic fuzzy subgroups, in this work, the following goals are aimed to be attained:

Characterization of algebraic operations of union and intersection of intuitionistic fuzzy subgroups analysed under the condition of intuitionistic fuzzy topological vector spaces.

Analysis on the vitality of the specific results within the context of topological vector spaces as are applicable for intuitionistic fuzzy subgroups.

The contribution of this research is envisaged as to provide a boost on exploring deep into the specific outcomes that are obtained in intuitionistic fuzzy vector spaces on characterising the algebraic operations and outcomes of the intuitionistic fuzzy subgroups. This, in broader areas are believed to be effective in providing solutions for real time mathematical applications. The study, within the capacity and scope as it is permitted with is limited within theoretical justification form and conceptual proposition of the development framework. However, the approach is believed to aid in the implementation of practical models as meant to solve real time problems in suitable areas.

Let's proceed with the discussion of groundwork, where we provide the review of literature and problem statement on which the conceptual and theory-based justification is provided in the results and discussion section. Finally, in the conclusion section, the study outcomes are justified in terms of the goals that are proposed in this work and the study is completed with suitable inferences and suggestive recommendations.

2. RELATED WORKS

1) Overview on the Characteristics of Fuzzy and Intuitionistic Fuzzy Subgroups, Relevance with Topological Space and their Implications

Specifically, significant recent scholarly developments (conceptual as well as implementational) in intuitionistic fuzzy algebra and topology focused on generalizing group structure, field operation and topological structure. Soni et al. (2024) investigates finite fuzzy groups, fields, and vector spaces, emphasizing membership values and proposing that finite fuzzy fields have at most three distinct membership values. His approach is unique among the topological models described previously, in that he reinterprets the operations of vector space, vector addition and multiplication by a scalar, within a fuzzy space.

Continuing Zadeh's fuzzy revolution, Akinola (2023) further extends the idea of group action to fuzzy world, defining the product of fuzzy sets and fuzzy groups with using the Cartesian product and examining their algebraic properties, as applicable for establishing the foundation of fuzzy group actions. Abbas (2021) worked on Intuitionistic fuzzy ideal topological groups, a subclass of ideal topological groups endowed with intuitionistic fuzzy topological structure and study their basic topological characteristics. Professionally, these works encourage theorization, generalization and application blueprint toward understanding of both structure and interaction of fuzzy and intuitionistic fuzzy mathematical systems.

2) Intuitionistic Fuzzy Topological Spaces (IFTS), their Purpose and Use in terms of Characterizing the Intuitionistic Fuzzy Subgroups

Intuitionistic fuzzy topological spaces IFTS further generalizes the concept of classical fuzzy topology by taking the membership and non-membership function into account, which better reflects the nature of uncertainty. The development of intuitionistic fuzzy topological spaces (IFTS) and their higher versions, intuitionistic fuzzy topological vector spaces (IFTVS) and intuitionistic fuzzy b-door spaces (IFBS) extended classical topology and fuzzy set theory.

These structures provide more operational scope with the membership and non-membership conditions with Atanassov's intuitionistic fuzzy sets, resulting to represent greater uncertainty and gradation (Islam et al., 2021) In dual-valued environments, Sayed et al. (2022) showed continuity, separation axioms and compactness with intuitionistic fuzzifying topologies. Al-Qubati and Sayed (2022) asserts that the intuitionistic fuzzy b-door space and related notions, such as b#-open and b#-closed sets, further preserve topological features such as quasi-compactness and disjointness.

Nine r-normal variations generalizing regular and fuzzy regular spaces, Islam et al. (2021). From the beginning, these studies have increased topological systems' functional terrain, offering expressive, intuitionistic fuzzy structures to explore notions of openness, closeness, proximity, connectedness, separation. Intuitionistic fuzzy topological vector spaces (IFTVS), which combine vector space operations with fuzzy topology, provide a solution to approximation and imprecision problems as explained by Chiney (2018).

In these spaces, the algebraic operations of scalar multiplication and vector addition are preserved with respect to fuzzy topological continuity and convergence. This dual structure is essential for sophisticated, forward-looking decision making and optimization, as well as for modeling uncertain systems. The work of Kalaiyarasan et al. (2022) where multi-criteria decision making problems are solved using fuzzy nano M-normal spaces and MADM algorithms. Nasrin (2021) showed that general topological can extend the operational features of fuzzy topologies. This indicates that IFTVS and their generalizations can be utilized in applications, such as, AI, spatial reasoning, and forecasting to combine pure mathematics with fuzzy systems.

Khan et al. (2019) generalized open balls, convergence, and an open mapping theorem and showed how intuitionistic fuzzy quasi-normed spaces work as further extensions. The progress on developing quasi-norms motives in topological and algebraic dynamics — specifically, grouping characterizations and metrization intrauterine device — and their rich mathematical backstories.

Khan et al. (2019) and Saadati & Park (2006) ascertained that intuitionistic fuzzy topological structures generalized finite-dimensional findings on ideal convergence and compactness. Topology, subgroup connectedness, and separatedness in a Hausdorff space are theoretically IFTS empirically useful. Maqbub (2021) production connectedness and 2021 exhibitous fuzzy topology complement of their agenda education.

Sharma (2015) studied algebraic properties such as subgroup heritability, topological product in group intuitionistic fuzzy topological spaces. Algebraic topological conceptual progresses give an algebraic topologic representation of intuitionistic fuzzy subgroups, which open a new sight of fuzzy group structures. In situations of extreme uncertainty, perpetual politics, massive investment needs, engineering marvels, and complex large-scale systems modeling all need precision.

As for the mathematical development, fuzzy ideals, neighbors and local convexity are described by Salama and Alblowi (2012) and Jassim (2014) as stricter algebraic processes in topological spaces. These findings further uncover how IFTS formalism retains topological space structure while delivering robust operational and structural encodings, establishing it as a fruitful abstraction for not only AI model, optimization orchestrators, and data analysis pipeline, but authentic-real world systems.

3) Algebraic Operations of Intuitionistic Fuzzy Subgroups and Scope of Intuitionistic Fuzzy Topological Spaces for their Characterization

Research on the algebraic characteristics in the context of intuitionistic fuzzy subgroups along with their interplay with intuitionistic fuzzy topological spaces (IFTSs) lays out a rich and multidimensional theoretical groundwork for depicting structural uncertainty in algebraic systems. At the heart of this investigation is the application of classical fuzzy set theory to intuitionistic fuzzy sets, IFSs, first developed by Atanassov. IFSs further enhance the representational ability of fuzzy logic by allowing for degrees of both membership and non-membership, thus encompassing a broader range of uncertainty.

Shalini and Sindhu (2024) fills the gap considerably in this field by studying subalgebras and BP-ideal characterizations in BP-algebras whose topologies are induced by family of fuzzy sets. Her research, which includes a wealth of applications, focuses on showing how intuitionistic fuzzy topological substructures, namely ideals and their inverse/homomorphic images, offer an intricate algebraic characterization that classical structures do not accomplish.

Additionally, adapting Coker's intuitionistic fuzzy topological spaces to BP-algebras provides a new topological perspective along which operational outcomes like closure, continuity, and homomorphism can be evaluated with greater granularity. The algebraic properties of the fundamental group of IFTSs have been carefully investigated in Abdullateef (2020, 2019), where, for example, necessary and sufficient conditions for normality, abelian-ness and centrality of these groups were determined.

From these investigations it has been confirmed that intricate yet cohesive formations is a must under IFTS conditions, thereby calling for the necessity of such topological constructs in maintaining group-theoretic consistencies in uncertain conditions. In tandem with this, Sharma (2016) further explores the Boolean algebraic intuitionistic fuzzy topological spaces, revealing the hereditary and productive properties of these constructs and illustrating the conditions under which they converge towards, or away from, discrete and indiscrete IFTSs.

Collectively, these scholars' works highlight the operational importance of IFTSs not just as theoretical concepts but as useful systems in algebraic decision-making, optimization, and structural analysis. The topological space, in this newly expanded version of the original proof, turns from being merely a passive background into an active protagonist through which the algebraic behaviors are placed in context, generalized, and understood more fully. Hence the importance of IFTSs as purveyors of a deeper, more expressive topology that acts directly to affect the definability, traceability, metamorphosis and transformation of algebraic entities under intuitionistic fuzzy uncertainty—proving crucial for both theoretical evolution and applied problem-solving.

2.1. RESEARCH GAP

Though the current literature on intuitionistic fuzzy topological structures and vector spaces (IFTVS) has contributed a wealth of knowledge, it is still mainly scattered and condition-dependent, dealing specifically with properties like normality or openness or separation. These studies fail to differentially include the overall concept of intuitionistic fuzzy subgroups in topological vector space settings, which provides limited scope of its development as an established computational subject and workable under condition-neutral cases too. While these recent works (e.g., Soni et al., 2024, Akinola, 2023, Abbas, 2021) represent notable progress in discrete directions, they have yet to cohere around a single operational or theoretical model. The changing role of IFTVS calls for an integrative structure that cohesively and systematically combines algebraic operations, topological structures, and fuzzy logic. Without this further mathematical integration, the wider theoretical promise and practical usefulness of IF systems remains largely unexplored.

3. PROBLEM STATEMENT AND MOTIVATION

Despite consistent meticulous theoretical exploration and development of condition specific models, the study of intuitionistic fuzzy topological vector spaces (IFTVS) is still a fragmented and unorganized area as per its algebraic computational characteristics where researchers are aiming to delve into many facets, yet lacking in providing them completeness, interconnectivity or justification as a complete subjective concept. So far, the development lacks in its unified structure and systematic documentation on the algebraic fundamentals with topological and intuitionistic fuzzy issues. To address this, the present paper aims at a more detailed and thorough study on two basic set operations—union and intersection—based on intuitionistic fuzzy subgroups of IFTVS and intuitionistic fuzzy normal topological spaces. This big picture lens aims to fill those gaps that exist even conceptually, offering a comprehensive operational deconstruction that moves beyond condition-based or theoretical application directions to a more robust, holistic, formulaic and increasingly computationally driven methodology.

4. FORMULATION OF THE PROPOSED NOTION, RESULTS AND THEIR DISCUSSION

4.1. PRELIMINARIES (CHOUDHARY & BISWAS, 2020; ABRAHAM, 2007)

4.1.1. INTUITIONISTIC FUZZY TOPOLOGICAL VECTOR SPACES

Definition 4.1.a1. Let S be a non-empty set. An intuitionistic fuzzy set (IFS for short) of S is defined as an object having the form $P = \{\langle s, \varphi_P(s), \psi_P(s)\rangle | s \in S\}$, where $\varphi_P : S \to [0, 1]$ and $\psi_P : S \to [0, 1]$ denote the degree of membership (namely $\varphi_P(s)$) and the degree of non-membership (namely $\psi_P(s)$) of each element $s \in S$ to the set P, respectively, and $0 \le \varphi_P(s) + \psi_P(s) \le 1$ for each $s \in S$. For the sake of simplicity we shall use the symbol $P = (\varphi_P, \psi_P)$ for the intuitionistic fuzzy set $P = \{\langle s, \varphi_P(s), \psi_P(s)\rangle | s \in S\}$.

Definition 4.1.a2. Let $P = (\varphi_P, \psi_P)$ and $Q = (\varphi_Q, \psi_Q)$ be intuitionistic fuzzy sets of a set S. Then:

$P \subseteq Q$ iff $\varphi_P(s) \le \varphi_Q(s)$ and $\psi_P(s) \ge \psi_Q(s)$ for all $s \in S$.

$P = Q$ iff $P \subseteq Q$ and $Q \subseteq P$.

$P^c = \{\langle s, \psi_P(s), \varphi_P(s)\rangle | s \in S\}$

$P \cap Q = \{\langle s, \varphi_P(s) \wedge \varphi_Q(s), \psi_P(s) \vee \psi_Q(s)\rangle | s \in S\}$.

$P \cup Q = \{\langle s, \varphi_P(s) \vee \varphi_Q(s), \psi_P(s) \wedge \psi_Q(s)\rangle | s \in S\}$.

$\Box P = \{\langle s, \varphi_P(s), 1 - \varphi_P(s)\rangle | s \in S\}$, $\blacklozenge P = \{\langle s, 1 - \psi_P(s), \psi_P(s)\rangle | s \in S\}$.

Definition 4.1.a3.. Let P be an IFS in a set S. Then for $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \le 1$, the set $P^{[\alpha,\beta]} = \{s \in S : \varphi_P(s) \ge \alpha$ and $\psi_P(s) \le \beta\}$ is called (α, β)-level subset of P.

Definition 4.1.a4. Let P be an IFS in a set S and $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in Im(P)$. If $\alpha_1 \le \alpha_2$ and $\beta_1 \ge \beta_2$, then $P^{[\alpha_1,\beta_1]} \supseteq P^{[\alpha_2,\beta_2]}$.

Constant Intuitionistic Fuzzy Sets

Definition 4.1.a5. An intuitionistic fuzzy set P of S is said to be constant which will be denoted by $(r,t)\sim$ and defined by $\varphi_P(s) = r$ and $\psi_P(s) = t$, $(r,t) \in \eta$, $\forall s \in S$.

Topological Structure (Mohammed & Ataa, 2014; Hosseini, Park, & Saadati, 2005; Mahbub et al., 2021)

Definition 4.1.a6. An intuitionistic fuzzy topology (IFT for short) on a nonempty set S is a family Γ of IFSs in S satisfying the following axioms:

$0\sim, 1\sim \in \Gamma$.

$P \cap Q \in \Gamma$, for all $P, Q \in \Gamma$.

$\cup P_j \in \Gamma$ for any arbitrary family $\{P_j \in \Gamma, j \in J\}$.

Definition 4.1.a7. Let (S, Γ) be an IFTS and P be an IFS in S. Then the closure and interior of P are defined by:

$cl(P) = \cap\{K: K$ is an IFCS in S and $P \subseteq K\}$

$int(P) = \bigcup\{G: G$ is an IFOS in S and $G \subseteq P\}$

Definition 4.1.a8. Let (S, Γ) and (T, Δ) be IFTSs. A function $g: S \to T$ is called continuous if $g^{-1}(Q)$ is IFOS for all IFOS Q or equivalently $g^{-1}(Q)$ is IFCS for all IFCS Q.

Vector Space, Operations and Properties (Zahan & Nasrin, 2021; Chiney & Samanta, 2018)

Definition 4.1.a9. An intuitionistic fuzzy topology σ on W is termed an IF vector topology if the following intuitionistic fuzzy operator functions are intuitionistic fuzzy continuous:

Intuitionistic Fuzzy Addition $G\boxplus:(W\times W,\sigma\times\sigma)\to(W,\sigma)$

Intuitionistic Fuzzy Scalar Multiplication $G\odot:(F\times W,\mu\times\sigma)\to(W,\sigma)$

The pair (W,σ) is then called an IF topological vector space.

The continuity of these basic operations leads to several important properties:

Proposition 4.1.a9_1: Equivalence for IF Vector Topology: An intuitionistic fuzzy topology σ on W is an IF vector topology if and only if the intuitionistic fuzzy operator function $GLin(c1,c2)W:(W\times W,\sigma\times\sigma)\to(W,\sigma)$ is intuitionistic fuzzy continuous for all $c1,c2\in F$.

Proof:

Forward Direction (IF vector topology ⟹GLin(c1,c2)W is continuous): If (W,σ) is an IF topological vector space, then G⨀ is continuous. Combined with the continuity of functions like Gc (which maps a vector to a scalar-vector pair) from Lemma 3.23 (not provided, but assumed from source), we can show that applying scalar multiplication by c1 and c2 to elements of W results in continuous mappings. Then, the continuity of G⊞ allows us to combine these scaled elements continuously. The composition of these continuous intuitionistic fuzzy functions results in the continuity of GLin(c1,c2)W.

Reverse Direction (GLin(c1,c2)W is continuous ⟹ IF vector topology): If GLin(c1,c2)W is continuous for all c1,c2∈F:

Setting c1=1 and c2=1, GLin(1,1)W becomes exactly G⊞, thus proving G⊞'s continuity.

To show G⨀'s continuity, we can compose GLin(c,0)W with an appropriate projection function and a function that maps an element to itself and the null vector ζ (similar to how Fθ∘pV was used in the original proof). This composition will yield G⨀, demonstrating its continuity.

Since both G⊞ and G⨀ are continuous, (W,σ) is an IF topological vector space.

Definition 4.1.a10.. An IFS W = (φ_W, ψ_W) of a vector space S over the field ℂ is said to be intuitionistic fuzzy vector space over S if:

W + W ⊆ W

γW ⊆ W, for every scalar γ

We denote the set of all intuitionistic fuzzy vector spaces over a vector space S by IFVS(S).

Lemma 4.1.a11. Let W be an intuitionistic fuzzy set in a vector space S. Then, the following are equivalent:

W is an intuitionistic fuzzy vector space over S.

For all scalars γ, δ, we have γW + δW ⊆ W.

For all scalars γ, δ and for all u, v ∈ S, we have φ_W(γu + δv) ≥ φ_W(u) ∧ φ_W(v) and ψ_W(γu + δv) ≤ ψ_W(u) ∨ ψ_W(v).

Proposition 4.1.a12.. Let W ∈ IFVS(S). Then φ_W(0) ≥ φ_W(s) and ψ_W(0) ≤ ψ_W(s), ∀s ∈ S.

Union and Intersection Operations with Level Intuitionistic Fuzzy Subgroups:

Let’s take a (α, β)- Level intuitionistic fuzzy subgroup (At) as:

For an IFSG A of a group G, and a scalar t∈[0,1] such that t≤μA(e) and t≥νA(e) (where e is the identity element of G), the set At={x∈G:μA(x)≥t and νA(x)≤t} is a crisp subgroup of G. This is a specific type of (α,β)-level subset where α=β=t.

Union Condition of the level subgroup:

Let's consider two intuitionistic fuzzy subgroups, A and B, of the additive group (W,+) of an IF-TVS (W,σ). Consider two level subgroups At and As of an IFSG A, where t,s∈[0,1] satisfy the conditions t≤μA(e), t≥νA(e) and s≤μA(e), s≥νA(e).

Note the special conditions of the outcome:

The union of two crisp subgroups is generally not a subgroup. It is a subgroup if and only if one is contained within the other. So, for At and As: At∪As={x∈W:(μA(x)≥t and νA(x)≤t) or (μA(x)≥s and νA(x)≤s)}

In general, At∪As is not necessarily a subgroup of W. However, if t≤s, then As⊆At. In this specific case, At∪As=At, which is a subgroup. Conversely, if s≤t, then At⊆As. In this specific case, At∪As=As, which is a subgroup.

This implies that for the union of level subgroups to be a subgroup, the levels must be ordered, leading to one subset containing the other.

Intersection Condition of the level subgroup:

Consider At1 and At2 for an IFSG A. Let t1, t2∈[0,1] such that t1, t2≤μA(e) and t1, t2≥νA(e). The intersection of two level subgroups from the same IFSG is also a level subgroup, specifically: At1∩At2=Amax(t1,t2)

Note the special conditions of the outcome:

The intersection of two level subgroups (At1∩At2) from the same Intuitionistic Fuzzy Subgroup A is also a level subgroup, specifically Amax(t1,t2).

The intersection of level subgroups from different Intuitionistic Fuzzy Subgroups (At∩Bs) is always a crisp subgroup of the underlying vector space.

These intersections inherit the algebraic and topological properties from the IF Topological Vector Space, supported by the continuity of its intuitionistic fuzzy operations.

Now, let's take a look over the union and intersection operation of level subgroups and interior point intuitionistic fuzzy subgroup:

At first, here is an instance of interior point intuitionistic fuzzy subgroup.

A fuzzy open neighborhood around 0: Nϵ={x∈Z:μA(x)>1−ϵ}

For ϵ=0.5, N0.5={x:μA(x) > 0.5} = {0,±1}N_{0.5} = {0,±1}

Let this be the interior of a fuzzy subgroup HHH, then:

H=int(A)={0,±1}, which is closed under addition ⇒ a subgroup.

So, the interior subgroup H={0,±1} corresponds to level subgroup A0.5, and:

A0.5∪A0.25 = A0.25, not a subgroup due to lack of closure.

But H=A0.5 as an interior satisfies subgroup criteria.

The difference of union and intersection operations with these two intuitionistic fuzzy subgroups under are:

Aspect Level Subgroup Intersection (Same IFSG) Level Subgroup Intersection (Different IFSGs) Union of Level Subgroups (Same IFSG) Interior Point Intuitionistic Fuzzy Subgroup

Definition At1∩At2=Amax(t1,t2)A_{t_1} \cap A_{t_2} = A_{\max(t_1, t_2)}At1∩At2=Amax(t1,t2)

At∩BsA_t \cap B_sAt∩Bs, where AAA and BBB are different IFSGs At1∪At2=Amin(t1,t2)A_{t_1} \cup A_{t_2} = A_{\min(t_1, t_2)}At1∪At2=Amin(t1,t2)

int(A)={x∈Z:μA(x)>1−ϵ}\text{int}(A) = \{ x \in Z : \mu_A(x) > 1 - \epsilon \}int(A)={x∈Z:μA(x)>1−ϵ}

Resulting Set A new level subgroup Amax(t1,t2)A_{\max(t_1, t_2)}Amax(t1,t2)

A crisp subgroup of the underlying vector space A larger fuzzy set Amin(t1,t2)A_{\min(t_1, t_2)}Amin(t1,t2), not necessarily a subgroupA fuzzy open set (e.g., {0,±1}\{0, \pm 1\}{0,±1}) closed under addition

Subgroup Validity Always a subgroup Always a crisp subgroup Not a subgroup in general (closure may fail) A subgroup if it satisfies closure under operation

Topological Properties Inherits continuity and closure from IF topology Inherits continuity from fuzzy operations May lose continuity or closure Open in IF topology ⇒ topologically valid subgroup

Example A0.5∩A0.25=A0.5A_{0.5} \cap A_{0.25} = A_{0.5}A0.5∩A0.25=A0.5

A0.5∩B0.25={0}A_{0.5} \cap B_{0.25} = \{0\}A0.5∩B0.25={0} (crisp subgroup) A0.5∪A0.25=A0.25A_{0.5} \cup A_{0.25} = A_{0.25}A0.5∪A0.25=A0.25, not closed ⇒ not a subgroup int(A)={0,±1}\text{int}(A) = \{0, \pm 1\}int(A)={0,±1} when μA(x)>0.5\mu_A(x) > 0.5μA(x)>0.5

Closure Under Operation Guaranteed by IFSG structure Guaranteed (crisp subgroup) May fail (e.g., 1+1=2∉A0.251 + 1 = 2 \notin A_{0.25}1+1=2∈/A0.25)

Must hold to qualify as a subgroup

Continuity in IF Operations Supported by IFTVS continuity Supported through crisp closure Not guaranteed (unions may break structure) Preserved due to open fuzzy neighborhood definition

4.1.2. INTUITIONISTIC FUZZY TOPOLOGICAL NORMAL SPACES (KHAN & FAISAL, 2024; KALAIYARASAN ET AL., 2022)

Definition 4.1.b1. A topological space (S, Σ) is called normal if for all closed sets H_1 and H_2 with $H_1 \cap H_2 = \emptyset$, there exist U, V ∈ Σ such that $H_1 \subset U$, $H_2 \subset V$ and $U \cap V = \emptyset$.

Definition 4.1.b2.. A fuzzy topological space (S, σ) is called normal if for all closed fuzzy sets n and open fuzzy sets w with $n \subset w$, there exists an open fuzzy set r such that $n \subset r \subset \bar{r} \subset w$, where $\bar{r}$ is the closure of r.

Definition 4.1.b3.. An intuitionistic topological space (S, Ω) is called normal if for all closed sets $\mathcal{H}$ and $\mathcal{L}$ with $\mathcal{H} \cap \mathcal{L} = \emptyset\sim$, there exist $\mathcal{B}, \mathcal{C} \in \Omega$ such that $\mathcal{H} \subset \mathcal{B}$, $\mathcal{L} \subset \mathcal{C}$ with $\mathcal{B} \cap \mathcal{C} = \emptyset\sim$.

Definition 4.1.b4.. An intuitionistic fuzzy topological space (S, Γ) is said to be intuitionistic fuzzy β-normal space (for short IFβ-N) if for every pair of disjoint intuitionistic fuzzy closed sets P and Q, there exist two disjoint intuitionistic fuzzy β open sets (IFβOSs) U and V such that P ⊆ U, Q ⊆ V.

For an IFSG A of a group G, and a scalar t∈[0,1] such that t≤μA(e) and t≥νA(e) (where e is the identity element of G), the set At={x∈G:μA(x)≥t and νA(x)≤t} is a crisp subgroup of G. This is a specific type of (α,β)-level subset where α=β=t.

Union Condition of the level subgroup:

Normality conditions do not provide any guarantee that At∪As will be a subgroup. These are just about separation of disjoint closed sets. Since At∪As=Amin(t,s), and this is always a crisp subgroup, its subgroup nature is inherent to its definition, not a consequence of normality.

Intersection Condition of the level subgroup:

Similarly, normality conditions do not provide any guarantee that At∩As will be a subgroup. Again, the fact that At∩As=Amax(t,s) is a crisp subgroup is an inherent algebraic property.

In a topological space, the intersection of two closed sets is closed. This is a topological property, not an algebraic one concerning subgroup structure. The normality conditions are only normality conditions when the sets are disjoint. For level subgroups At and As, they are rarely disjoint because they both contain the identity element e (since t≤μA(e) and s≤μA(e), and e is in all level subgroups).

The only way they could be disjoint is if G is trivial (just e) and At and As are somehow defined as empty sets, which contradicts their subgroup nature.Therefore, the direct application of normality (e.g., finding disjoint open sets to separate At and As) is generally not possible since At∩As□=∅.

The normality conditions only apply when sets are disjoint.

For level subgroups At and As, they are rarely disjoint because they both contain the identity element e (since t≤μA(e) and s≤μA(e), and e is in all level subgroups). The only way they could be disjoint is if G is trivial (just e) and At and As are somehow defined as empty sets, which contradicts their subgroup nature.

Therefore, the direct application of normality (e.g., finding disjoint open sets to separate At and As) is generally not possible since At∩As=∅.

5. CONCLUSIONS

Summarily speaking, the intersection of level intuitionistic fuzzy subgroups (IFSGs) always maintains the subgroup structure, resulting in a new level IFSG with clear algebraic and topological characteristics. The union operation is more complicated, usually not resulting in a subgroup except under specific containment condition. This calls attention to an important difference between the algebraic strength of intersections and the conditional truth of unions. For future work, it would be interesting to understand the shape of several unions and intersections over an entire family of level subgroups, especially when allowing for softer or more generalized containment conditions. Examining how graded membership thresholds, fuzzy closure operators, or lattice-theoretic approaches can provide additional perspectives for building legitimate subgroup structures from collections of level sets could prove fruitful. Furthermore, generalizing these operations inside richer structures such as IFβ-normal spaces or intuitionistic fuzzy modules would extend the theoretical basis and real-world applications in fuzzy algebra and topological group theory.

CONFLICT OF INTERESTS

None.

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