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APPROXIMATION OF FIXED POINT FOR F ITERATIVE ALGORITHM AND SOLUTION OF A DELAY DIFFERENTIAL EQUATION

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ABSTRACT

The paper reports convergence, stability and data dependence results for the operators satisfying contractive conditions and contractive condition of rational expression using F iteration scheme in Banach space. With the help of suitable numerical examples, it is claimed that F iteration process is more efficient than many other iterative schemes available in literature. As an application, we have proposed solution to a delay differential equation. Our results are new and extends and improves many corresponding results available in the literature.
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1. INTRODUCTION

Over the years, researchers are trying to develop faster iterative scheme for different operators and sill it is a challenging question for them to find an iterative scheme with better efficiency. In the present work, we have compared the efficiency of various iterative schemes available in the literature using different control sequences and different initial points.

It is well known that finding solution to a nonlinear equation and approximating fixed points of a corresponding contractive type operator have close association. It creates interest of researchers towards approximating the fixed point of these operators. Famous Picard [13] iterative scheme is generally used to approximate these operators satisfying the following condition (contractive condition):

$$\|T_x - Ty\| \leq \delta \parallel x, y \parallel \qquad (1.1)$$

where T:H→H is an operator, δ∈(0,1),x,y∈H and H is a non empty subset of a Banach space X. A point x∈H is called fixed point of the mapping T if Tx=x.

It is the drawback of Picard's iteration that it fails to converge to fixed point of nonexpansive mappings. To overcome it, Mann [7] introduced the following iterative scheme:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n \,. \qquad (1.2)\text{ "}$$

In [6], Ishikawa produced examples to claim that Mann's iteration process fails to converge to fixed points of pseudo-contractive operator and defined the following iterative scheme to overcome it.

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n. \end{aligned} \qquad (1.3)$$

Noor [8], introduced the following iterative scheme to give solution to variational inequalities:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ z_n &= (1-\gamma_n)x_n + r_n T x_n. \end{aligned} \qquad (1.4)$$

In [14], Khan introduced the Picard Mann hybrid iterative scheme for nonexpansive mappings as follows:

$$\begin{aligned} x_{n+1} &= T y_n \\ y_n &= (1-\alpha_n)x_n + d_n T x_n\,. \end{aligned} \qquad (1.5)$$

Khan [14], also claimed its efficiency than some of existing iterative schemes in the sense of Berinde [4] for contractive mappings.

Recently, Okeke and Abbas [15] introduced Picard-Krasnoselskii iterative process given by

$$\begin{aligned} x_{n+1} &= T y_n \\ y_n &= (1-\lambda)x_n + \lambda T(x_n). \end{aligned} \qquad (1.6)$$

They used this scheme to find solution of delay differential equation.

Ali and Ali [2], introduced the F iterative scheme for contraction mappings as follows:

$$\begin{aligned} z_n &= T\big((1-\alpha_n)x_n + \alpha_n T x_n\big) \\ y_n &= T z_n \\ x_{n+1} &= T y_n. \end{aligned} \qquad (1.7)$$

They showed the stability, better rate of convergence of this iterative procedure in the setting of generalized contractions. Some notable work in this direction is due to [9,10,18,19].

Definition 1.1[4]: Let T_1 T ̃:H→H be two operators. We say that T ̃ is an approximate of T if for all x,y∈H and for a fixed ε>0, we have

$$\| Tx - \tilde{T}x \| \leq \varepsilon. \qquad (1.8)\text{ ")}$$

Now we recall some lemmas which are important in proving the main results.

Lemma 1.2 [4] : Let δ is a real number such that 0⩽S<1 and {ε_n }_(n=0)^∞ is a sequence of positive numbers such that lim_(n→∞)ε_n=0, then for any sequence of positive numbers {u_n }_(n=0)^∞ satisfying u_(n+1)≤δu_n+ε,□( ) n=0,1,2,...

Then lim_(n→∞)u_n=0.

Lemma 1.3 [17]: Let {β_n }_(n=0)^∞ and {ρ_n }_(n=0)^∞ be nonnegative real sequences satisfying the following inequalities:

$$\beta_{n+1} \leq (1-\lambda_n)\beta_n + \rho_n,$$

where $\lambda_n \in (0,1)$ for all $n \geq n_1, \sum_{n=1}^{\infty} \lambda_n = \infty$ and $\frac{\rho_n}{\lambda_n} \to 0$ as $n \to \infty$ Then lim_(n→∞)β_n=0.

Lemma 1.4 [16]: Let $\{\beta_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists n_0∈N such that for all n≥n_0 the following inequality holds

$$\beta_{n+1} \leq (1-\mu_n)\beta_n + \mu_n\gamma_n,$$

where $\mu_n \in (0,1)$ for all $n \in N, \sum_{n=0}^{\infty}\mu_n = \infty$ and $\gamma_n \geq 0\ \forall N$.

Then the following inequality holds

$$0 \leqslant \limsup_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\gamma_n.$$

Ostrowski [12], introduced the concept of stability for iterative schemes. Generally, in the process of approximating fixed points, we consider appropriate sequence instead of original sequence due to rounding errors and numerical approximations of functions.

Definition 1.5 [12]: Let T be a self map defined on Banach space with fixed bunt p. Let {t_n } be any arbitrary sequence in X. Consider the iterative scheme x_(n+1)=f(T,x_n) for some function f, converging to P, is said to be stable w.r.t. T if and only if for

$$\varepsilon_n = \| t_{n+1} - f(T,t_n) \|, \text{ we have } \lim_{n\to\infty}\varepsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} t_n = P.$$

2. CONVERGENCE ANALYSIS OF F-ITERATION PROCESS IN BANACH SPACES

We begin the section by claiming that strong convergence of F-iteration process and its higher rate of convergence than Picard-Mann hybrid iteration process [14] and Picard- Krasnoselskii hybrid iterative process [15]. We also compare the rate of convergence of F-itertion process with some of the existing iteration schemes available in the literature.

Theorem 2.1: Let H be a nonempty closed convex subset of a Banach space E and T:H→H be a mapping satisfying the contractive condition (1.1). Let {x_n } be the iterative sequence generated by (1.7) with real sequence {α_n }_(n=0)^∞ in [0,1] satisfying ∑_(n=0)^∞α_n=∞. Then {x_n } converges strongly to a unique fixed point of T.

Proof: By consequences of Banach contraction principle the existence and uniqueness of fixed point p is decided. We need only to prove the convergence of iterative scheme (1.7) to fixed point p. Now

$$\begin{aligned}
\|z_n - p\| &= \| T\big((1-\alpha_n)x_n + \alpha_n T x_n\big) - p \| \\
&= \| T\big((1-\alpha_n)x_n + \alpha_n T x_n\big) - Tp \| \\
&\leq \delta\|(1-\alpha_n)x_n + \alpha_n T x_n - p\| \\
&\leq \delta\|(1-\alpha_n)(x_n - p) + \alpha_n(T x_n - p)\| \\
&\leq \delta[(1-\alpha_n)\|x_n - p\| + \alpha_n \cdot \delta\|x_n - p\|] \\
&\leq \delta[(1-\alpha_n + \alpha_n\delta)\|x_n - p\|] \\
&= \delta[(1-(1-\delta)\alpha_n)\|x_n - p\|] \\
&\leq \delta\big(1-\alpha_n(1-\delta)\big)\|x_n - p\|. \qquad (2.1)
\end{aligned}$$

Again using (1.1), (1.7) and (2.1) we have:

$$\begin{aligned}\|y_n - p\| &= \|Tz_n - p\| \\ &= \|Tz_n - Tp\| \\ &\le \delta\|z_n - p\| \\ &\le \delta^2\big(1 - \alpha_n(1-\delta)\big)\|x_n - p\|. \end{aligned} \tag{2.2}$$

Now using (1.1), (1.7) and (2.2) we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|Ty_n - Tp\| \\ &\le \delta\|y_n - p\| \\ &\le \delta^3\big(1 - \alpha_n(1-\delta)\big)\|x_n - p\|. \end{aligned} \tag{2.3}$$

Now $(1 - \alpha_n(1-\delta)) < 1$ and $\delta \in (0,1)$, we can write

$$\|x_{n+1} - p\| \le s^3(1 - \alpha_n(1-\delta))\|x_n - p\|$$
$$\|x_n - p\| \le \delta^3(1 - \alpha_{n-1}(1-\delta))\|x_{n-1} - p\|$$
$$\|x_{n-1} - p\| \le \delta^3\big(1 - \alpha_{n-2}(1-\delta)\big)\|x_{n-2} - p\|$$
$$\ldots\ldots\ldots\ldots\ldots$$

$$\| x_n - p \| \le \delta^3(1 - \alpha_1(1-\delta))\|x_1 - p\| . \tag{2.4}$$

From the above in above inequalities, we derive

$$\|x_{n+1} - p\| \le \delta^{3(n+1)}\|x_1 - p\| \prod_{i=1}^{n} \big(1 - \alpha_i(1-\delta)\big), \tag{2.5}$$

where $(1 - \alpha_i(1-\delta)) \in (0,1)$ since $\delta \in (0,1)$ and $\alpha_i \in [0,1]$ for all $i \in N$.

Now using this famous result from analysis that 1-x≤e^(-x) for all x∈[0,1] in (2.5), we obtain

$$\|x_{n+1} - p\| \le \frac{\delta^{3(n+1)}\|x_1 - p\|}{e^{(1-\delta)\sum_{i=1}^{n} d_i}} . \tag{2.6}$$

Hence, $\lim_{n\to\infty}\|x_{n+1} - p\| \le \left\{\frac{\|x_1 - p\|\delta^{3(n+1)}}{e^{(1-\delta)\sum_{i=1}^{n}\alpha_i}}\right\} \to 0$ as $n \to \infty$.

It means that $\lim_{n\to\infty}\|x_n - p\| = 0$. This completes the proof.

Theorem 2.2: Let H be a non empty closed convex subset of a Banach space E and T:H→H be a mapping satisfying the contractive condition (1.1). Let each of the iterative process (1.5), (1.6) and (1.7) converges to same fixed point P of T, where $\{\alpha_n\}_{n=0}^{\infty}$ and λ are such that $0 \le \lambda, \alpha_n < 1$ for all $n \in N$. Then iteration scheme (1.7) has higher rate of convergence than (1.5) and (1.6).

Proof: Using ([14], proposition 1), we have

$$\|u_{n+1} - p\| \leq [\delta(1-\delta)\alpha]^n \|u_1 - p\| \,. \quad (2.7)$$

Let

$$a_n = [\delta(1-(1-\delta)\alpha]^n \|u_1 - p\|. \quad (2.8)$$

Using ([15], proposition 2.1), we have

$$\|V_{n+1} - p\| \leq [\delta(1-(1-\delta)\alpha]^n \|v_1 - p\|. \quad (2.9)$$

Let $b_n = [\delta 1^{1-}(1-\delta)\alpha]^n \|v_1 - p\|.$ (2.10)

Using (2.3), we have

$$\|x_{n+1} - p\| \leq \delta^3(1-\alpha_n(1-\delta))\|x_n - p\|$$

$$\Rightarrow \|x_{n+1} - p\| \leq \left[\delta^3\left(1-\alpha(1-\delta)\right)\right]^n \|x_1 - p\|. \quad (2.11)$$

Let $c_n = \left[\delta^3\left(1-\alpha_n(1-\delta)\right)\right]^n \|x_1 - p\|.$ (2.12)

Using (2.12) and (2.10), we have

$$\frac{c_n}{b_n} = \frac{[\delta^3(1-(1-\delta)\alpha]^n \|x_1 - p\|}{[\delta(1-(1-\delta)\alpha]^n \|v_1 - p\|} = \frac{\delta^{3n}\|x_1 - p\|}{\|v_1 - p\|} \to 0 \text{ as } n \to \infty.$$

Hence, {x_n } converges faster to P than {v_n }.

Using (2.12) and (2.8), we have

$$\frac{c_n}{a_n} = \frac{[\delta^3(1-(1-\delta)\alpha]^n \|x_1 - p\|}{[\delta(1-\delta)\alpha)^n \ \|} \|u_1 - p\| = \frac{\delta^{3n}\|x_1 - p\|}{\|u_1 - p\|} \to 0 \text{ as } n \to \infty.$$

Hence, {x_n} converges faster to P than {u_n }.This completes the Proof.

Now, we support Theorem 2.2 with following numerical example:

Example 2.1: Let $H = [1,10]$ and T:H→H be an operator defined by $T(x) = \sqrt[3]{2x+4}$ for all x∈H. Let $\alpha_n = \lambda = \frac{1}{2}$ for each n∈N. Clearly, T is a contraction mapping with constant $\frac{1}{3\sqrt{4}}$. Also, the unique fixed point of the mapping T is 2. Table 2.1. shows the higher rate of convergence of iteration scheme (1.7) than (1.5) and (1.6).

Table 2.1: Values generated of various iterative schemes for mapping T of Example 2.1

Step	Iteration (1.7)	Iteration (1.5)	Iteration (1.6)
1.	5.00000000	5.0000000	5.00000000
2.	2.00681559913	2.25128435404	2.25128435407
3.	2.0001839755	2.02406896909	2.02406896909

4	2.00000004968	2.00236363938	2.0023636968
5.	2.0000000117	2.00022714115	2.00022714115
6	2.00000000003	2.00002208286	2.00002208286
7	2.00000000000	2.00000214694	2.00000021694
.	.	.	.
.	.	.	.
.	.	.	.
14	2.00000000000	2.00000000002	2.0000000002
15	2.00000000000	2.00000000000	2.0000000000

Example2.2: Let $H = [0,1]$ and $T: H \rightarrow H$ be an operator satisfying $T(x) = x/2$ for all x∈H. Clearly, 0 is the only fixed point of the operator T. Now, for $a_m = .70, b_m = .65, c_m = .90$, the values of iterative schemes M [9],Picard-S [5],Abbas[1],Noor[8],Ishikawa[6],Mann[7] and F iteration are given in Table 2.2.

Table 2.2: Iterative values obtained by various schemes for the operator T of Example 2.2

Steps		F	M	Picard-S	Abbas	Agarwal	Noor	Ishikawa	Mann
1		0.8000	0.8000	0.8000	0.8000	0.8000	0.8000	0.8000	0.8000
2		0.065	0.1300	0.1545	0.1964	0.3090	0.3880	0.4290	0.5200
3		0.0052	0.2112	0.0298	0.0482	0.1193	0.1882	0.2300	0.3380
4		0.0004	0.0034	0.0057	0.0118	0.0460	0.0913	0.1233	0.2197
5		0	0.0005	0.0011	0.0029	0.0178	0.0442	0.0661	0.1428
6		0	0	0.0002	0.0007	0.0068	0.0214	0.0354	0.0928
7		0	0	0	0.0001	0.0026	0.0104	0.0190	0.0603
8		0	0	0	0	0.0010	0.0050	0.0102	0.0392
9		0	0	0	0	0.0003	0.0024	0.0054	0.0254
10		0	0	0	0	0.0001	0.0011	0.0029	0.0165
11		0	0	0	0	0	0.0005	0.0015	0.0107
12		0	0	0	0	0	0.0002	0.0008	0.0070
13		0	0	0	0	0	0.0001	0.0004	0.0045
14		0	0	0	0	0	0	0.0002	0.0029
15		0	0	0	0	0	0	0.0001	0.0019
16		0	0	0	0	0	0	0	0.00012

Now we study the influence of initial points for various iteration schemes in obtaining the fixed point. Table 2.3, provides the number of iterations required to obtain fixed point for various initial points in Example 2.2.

Table 2.3: Number of iterations required to obtain fixed point for the operator T of Example 2.2

Initial Value	Agarwal iteration	Picard-S iteration	M- iteration	F-iteration
0. 2	10	6	5	4
0.4	11	6	5	4
0.6	12	7	6	5

0.8	12	7	6	5
1.0	12	7	6	5

Now, we prove the following convergence result using a contractive condition satisfying rational expression:

Theorem 2.4: Let H be a nonempty closed convex subset of a Banach space E and T:H→H be a mapping defined by the following rational expression

$$\| T_x - T_y \| \leq \frac{\psi(\|x-T_x\|)+a\|x-y\|}{1+M\|x-T_x\|} \qquad (2.13)$$

for all $x, y \in H, a \in [0,1), M \geq 0, \psi$ is a monotone increasing function with $\psi(0) = 0$. Let $\{x_n\}$ be the sequence generated by (1.7) where α_n<1 for all n∈N. Then iterative process (1.7) converges to unique fixed point of T.

Proof: First, we prove the convergence of iterative scheme.

Using (1.7) and (2.13) we have

$$\begin{aligned} \|z_n - p\| &= \| T\big((1-\alpha_n)x_n + \alpha_n T x_n\big) - p \| \\ &= \| p - T\big((1-\alpha_n)x_n + \alpha_n T x_n \| \\ &\leq \frac{\varphi \| p - Tp \|) + q \| p - \big((1-\alpha_n)x_n + \alpha_n T x_n\big)}{1 + M \| p - Tp \|} \end{aligned}$$

$$\begin{aligned} \|z_n - p\| &\leq a\|p - (1-\alpha_n)x_n + \alpha_n T x_n\| \\ &\leq a\big[(1-\alpha_n)\|x_n - \rho\| + \alpha_n \|T_{p_n - P\|}\|\big] \\ &\leqslant a\left[(1-\alpha_n)\|x_n - p\| + \alpha_n \left[\frac{\psi \| p - Tp \| \, q_a\|a_n - p\|}{1 + \|\| p - T\rho \|}\right]\right) \\ &\leq a(1-\alpha_n)\|x_n - p\| + ad_n(q\|x_n - p\|) \\ &\leq (a(1-\alpha_n) + a^2\alpha_n)\|x_n - p\| \\ &\leq \big(a(1-\alpha_n + a\alpha_n)\big)\|x_n - p\| \\ &\leqslant a(1-(1-a)\alpha_n)\big)\|x_n - p\|. \qquad (2.14) \end{aligned}$$

Again using (1.7) and (2.13), we have

$$\begin{aligned} \|y_n - p\| &= \|Tz_n - p\| \\ &= \|Tp - Tz_n\| \\ &\leqslant \frac{\psi(\| p - Tp \|) + q\|z_n - p\|}{1 + M \| p - Tp \|} \\ &\leq a\|z_n - p\| \\ &\leq a^2(1-(1-q)\alpha_n)\big)\|x_n - p\|. \qquad (2.15) \end{aligned}$$

Now

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| = \|Tp - Ty_n\| \\ &\leq \frac{\psi(\| p - Tp \|) + a\|y_n - p\|}{1 + M \| p - Tr \|} \\ &\leq q\|y_n - p\| \\ &\leqslant q^3(1-(1-q)\alpha_n)\big)\|x_n - p\|. \qquad (2.16) \end{aligned}$$

From (2.16), we can deduce that,

$$\|x_{n+1} - p\| \leqslant a^3(1-(1-a)a_n)\big)\|x_n - p\|$$

$$\|x_n - p\| \leq a^3(1-(1-a)\alpha_{n-1})\big)\|x_{n-1} - p\|$$

$$\|x_2 - p\| \leq a^3(1-(1-a)\alpha_1)\big)\|x_1 - p\|$$

Hence, $\|x_{n+1} - p\| \leq q^{3(n+1)}\|x_1 - p\| \prod_{i=1}^{n} \big(1 - \alpha_i(1-a)\big), \quad (2 \cdot 17)$

where $(1-\alpha_i(1-a)) \in (0,1)$ since a∈[0,1) and α_i∈[0,1] for all n∈N.

Now using the result $1-x \leq e^{-x}$ for all x∈[0,1] we obtain from (2.17)

$$\|x_{n+1}-p\| \leq \frac{a^{3(n+1)}\|x_1-p\|}{e^{(1-q)\sum_{i=1}^{n} \alpha_i}}.$$

Therefore $\lim_{n\to\infty} \|x_{n+1}-p\| \leq \frac{a^{3(n+1)}\|x_1-p\|}{e^{(1-q)\sum_{i=1}^{n} \alpha_i}} \to 0$ as $n \to \infty$.

It proves the convergence of sequence {x_n } to fixed point p. Now, we show that T has unique fixed point. Let if possible p^* is another fixed point of mapping T. Then we have

$$\begin{aligned} \| p-p*\| &= \| Tp-Tp*\|) \\ &\leq \frac{\psi \| p-Tp \| + a \| p-p*\|}{1+M \| p-Tp \|} \\ &\leq a \| p-p*\|. \end{aligned}$$

This implies that p=p*. It establishes the proof of theorem.

3. STABILITY RESULTS IN COMPLEX VALUED BANACH SPACES

Theorem 3.1: Let H be a nonempty closed convex subset of a Banach space E and T:H→H be a mapping defined by (1.1). Let for some $p \in F(T)$, the iterative scheme (1.7) converges to p, where $\sum_{n=0}^{\infty} r_n = \infty$ for each n∈N. Then F iteration process is T- stable.

Proof: Let $\{u_n\}_{n=1}^{\infty} \in H$ H be any arbitrary bounded sequence.

let $\varepsilon_n = \| u_{n+1} - T_{a_n} \|$, (3.1)

where $a_n = T((1-\alpha_n)u_n + \alpha_n T u_n)$.

Using (1.7) and (3.1) we have

$$\begin{aligned} \|u_{n+1}-P\| &\leq \| u_n - Ta_n \| + \| Ta_n - p \| \\ &\leq \varepsilon_n + \delta\|a_n - P\| \\ &\leq \varepsilon_n + \delta[\|T(1-\alpha)u_n + \alpha_n T u_n) - P\|] \\ &\leq \varepsilon_n + \delta[\delta^3(1-\alpha_n(1-\delta))\|u_n - P\|] \\ &\leq \varepsilon_n + \delta^4(1-\alpha_n(1-\delta))\|u_n - P\| . \qquad (3.2) \end{aligned}$$

Since α_n∈(0,1) for all n∈N and δ∈(0,1) we have (1-α_n (1-δ))<1. Hence, by lemma 1.3 and (3.2), we have lim_(n→∞)u_n=p. Conversely

$$\begin{aligned} \varepsilon_n &= \|u_{n+1} - Ta_n\| \\ &\leq \|u_{n+1} - p\| + \|p - Ta_n\| \\ &\leq \|u_{n+1} - p\| + \delta\|a_n - p\| \\ &\leq \|u_{n+1} - p\| + \delta \ \| T((1-\alpha_n)u_n + \alpha_n T_n) - p \| \\ &\leq \|u_{n+1} - p\| + \delta[\delta^3(1-\alpha_n(1-\delta))\|u_n - p\|] \\ &\leq \|u_{n+1} - p\| + \delta^4((1-\sin_n(1-\delta))\|u_n - p\|). \end{aligned}$$

Since δ^4 (||-α_n (1-δ))<1, we obtain

$$\varepsilon_n \leqslant \|u_{n+1} - p\| + \|u_n - p\|.$$

Taking limit n→∞ both sides, we have

$$\lim_{n\to\infty} \varepsilon_n = 0$$

It means that F-iteration is T-stable.

Example 3.2: Let H=[0,1] and T:H→H be a mapping defined by T(x)=x/2. Then T is a contraction mapping with constant 1/2. Clearly, 0 is the unique fixed point of this mapping. Suppose $\{u_n\} = \frac{1}{n}$, is any arbitrary sequence in H and $\alpha_n = \frac{1}{2}$ for each $n \in N$.. Then $\lim_{n\to\infty} u_n = 0$. Let

$$\varepsilon_n = |u_{n+1} - f(T, a_n) = |u_n - Ta_n|, \quad (3.3)$$

where $a_n = T(1-\alpha_n)u_n + \alpha_n T u_n)$.

Now
$$\begin{aligned}\varepsilon_n &= |u_{n+1} - Ta_n| \\ &= \left|u_{n+1} - \frac{a_n}{2}\right| \\ &= \left|u_{n+1} - \frac{(1-\alpha_n)u_n + \alpha_n T u_n}{4}\right| \\ &= \left|1u_{n+1} - \frac{(1-\alpha_n)u_n}{4} - \frac{\alpha_n u_n}{8}\right| \\ &= \left|\frac{1}{n+1} - \frac{1}{8n} - \frac{1}{16n}\right|.\end{aligned}$$

Hence, $\lim_{n\to\infty} \varepsilon_n = 0.$

Hence, F-iteration scheme is T-stable.

Theorem 3.2: Let H be a non empty closed convex subset of a Banach space E and T:H→H be a mapping given by the following rational expression:

$$\| Tx - Ty \| \leq \frac{\psi(\| x - Tx \|) + a \| x - y \|}{1 + M\|x - T_x\|}, \quad \forall x, y \in H, a \in [0,1), M > 0$$

where ψ is a monotone increasing function such that ψ(0)=0. let $\{x_n\}$ be the sequence generated by F iteration satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\alpha_n \leq \alpha \in (0,1)$ for each $n \in N$. This iteration scheme is T- stable.

Proof: Suppose $\{u_n\}_{n=1}^{\infty} CH$ CH be any arbitrary bounded sequence.

Put $\varepsilon_n = \|u_{n+1} - Tb_n\|,$

where $b_n = T\big((1-\alpha_n)u_n + \alpha_n T u_n\big)$.

Now

$$
\begin{aligned}
\|u_{n+1} - p\| &\le \|u_{n+1} - Tb_n\| + \|Tb_n - p\| \\
&= \|u_{n+1} - Tb_n\| + \|p - Tb_n\| \\
&\le \varepsilon_n + \frac{\psi(\| p - Tp \| + a\|b_n - p\|)}{1 + M \| p - Tp \|} \\
&= \varepsilon_n + a\|b_n - p\| \\
&= \varepsilon_n + a\Big(T\big((1-\alpha_n)u_n + \alpha_n T u_n\big) - p\Big) \\
&\le \varepsilon_n + a(a^3(1-(1-a)\alpha_n)\|k_n - p\|).
\end{aligned}
$$

(using the arguments similar to 2.16)

Now,

$$\|u_{n+1} - p\| \le \varepsilon_n + (a^4((1-(1-a)\alpha_n)\|u_n - p\|). \qquad (3.4)$$

Since $\alpha_n \le \alpha \in (0,1)$ for all $n \in N$ and a $a \in [0,1)$ we have $a^4(1-(1-q)\alpha_n) < 1$..

Hence, $\lim_{n\to\infty}\|u_n - p\| = 0 \Rightarrow \lim_{n\to\infty} u_n = p$.

Conversely,

$$
\begin{aligned}
\varepsilon_n &= \|u_{n+1} - Tb_n\| \\
&\le \|u_{n+1} - p\| + \|p - Tb_n\| \\
&\le \|u_{n+1} - p\| + \frac{\psi(\| p - Tp \|) + q\|b_n - p\|}{1 + M(\| p - Tp \|)} \\
&\le \|u_{n+1} - p\| + a\|b_n - p\| \\
&\le \|u_{n+1} - p\| + a\big[\| T\big((1-\alpha_n)u_n + \alpha_n T u_n\big) - p \|\big].
\end{aligned}
$$

Using arguments similar to 2.16, we obtain

$$\varepsilon_n \le \|u_{n+1} - p\| + a^4(1-(1-a)\alpha_n)\|u_n - p\|$$

$$\le a^4(1-(1-a)\alpha_n) < 1. \qquad (3.5)$$

Taking limit n →∞, we have lim_(n→∞)ε_n=0. This completes the proof.

4. DATA DEPENDENCE RESULT

Theorem 4.1: Let T ˜ be an approximate operator of contraction T:H→H. defined by (1.1). Let $\{x_n\}_{n=1}^{\infty}$ be the iterative scheme generated by (1.7) for T. Consider the iterative sequence $\{\tilde{x}_n\}_{n=1}^{\infty}$ as follows:

$$
\begin{cases}
\tilde{z}_n = \tilde{T}\left((1-\alpha_n)\tilde{x}_n + \alpha_n \tilde{T}_n \tilde{x}_n\right) \\
\tilde{y}_n = \tilde{T}\tilde{z}_n \\
\tilde{x}_{n+1} = \tilde{T}\tilde{y}_n, n \in N ,
\end{cases} \qquad (4.1)
$$

with real sequence $\{\alpha_n\}_{n=1}^{\infty}$ in [0,1] satisfying the following conditions

(i) $\frac{1}{2} \leq \alpha_{n_}$n for all n+N

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$

If $Tp = p$ ˜=$\tilde{T}\tilde{p} = \tilde{p}$such that $\lim_{n\to\infty} \tilde{u}_n = \tilde{p}$, then

we have$\| p - \tilde{p} \| \leq \frac{4\varepsilon}{1-\delta}$ where $\varepsilon > 0$ is a fixed number.

Proof: Using (1.7), (1.1) and (4.1) we have

$$\|z_n - \tilde{z}_n\| = \| T\big((1-\alpha_n)x_n + \alpha_n T x_n\big) - \tilde{T}\left((1-\alpha_n)\check{x}_n + \alpha_n \check{T}\check{x}_n\right) \|.$$

Let $b_n = (1-\alpha_n)x_n + \alpha_n T x_n$,

$\tilde{b}_n = (1-\alpha_n)\tilde{x}_n + \alpha_n \tilde{T}\tilde{x}_n$.

$$\|z_n - \tilde{z}_n\| = \| Tb_n - \tilde{T}\tilde{b_n} \| = \| Tb_n - T\tilde{b}_n + T\tilde{b}_n - \tilde{T}\check{b}_n \|$$
$$\leq \| Tb_n - T\tilde{b}_n \| + \| T\tilde{b}_n - \tilde{b}_{b_n} \|$$
$$\leq s \| b_n - \tilde{b}_n \| + \varepsilon$$
$$\leq \delta\big[\| (1-\alpha_h)x_n + \alpha_h T x_n - (1-\alpha_h)x_h - \alpha_h \tilde{T}\check{x}_n \|\big] + \varepsilon.$$

Since δ<1, we have

$$\begin{aligned}\|z_n - \tilde{z}_n\| &\leq (1-\alpha_n)\|x_n - \tilde{x}_n\| + \alpha_n\|Tx_n - \tilde{T}\tilde{x}_n\| + \varepsilon \\ &\leq (1-\alpha_n)\|x_n - \tilde{x}_n\| + \alpha_n\|Tx_n - T\tilde{x}_n + T\tilde{x}_n - \tilde{T}\tilde{x}_n\| + \varepsilon \\ &\leq (1-\alpha_n)\|x_n - \tilde{x}_n\| + \alpha_n\delta\|x_n - \tilde{x}_n\| + \alpha_n\varepsilon + \varepsilon \\ &\leq (1-\alpha_n + \alpha_n\delta)\|x_n - \tilde{x}_n\| + 2\varepsilon \\ &\leq (1-(1-\delta)\alpha_n\|\alpha_n - \tilde{x}_n\| + 2\varepsilon.\end{aligned}$$

Now using ,(1.1) we have

$$\begin{aligned}\|y_n - \tilde{y}_n\| &= \| T_{z_n} - \tilde{T}\tilde{z}_n \| \\ &= \| Tz_n - \tilde{T}\tilde{z}_n + T\tilde{z}_n - \check{T}\tau_n \| \\ &\leq \|Tz_n - T\tilde{z}_n\| + \| T\tilde{z}_n - \tilde{T}\tilde{z}_n \| \\ &\leq \|z_n - \tilde{z}_n\| + \varepsilon. \qquad (4.3)\end{aligned}$$

Using (4.2) in (4.3), we have

$$\begin{aligned}\|y_n - \tilde{y}_n\| &\leq \delta[1-(1-\delta)\alpha_n\|x_n - y_n\| + 2\varepsilon] + \varepsilon \\ &\leq (1-(1-\delta)\alpha_n\|x_n - \check{x}_n\| + 3\varepsilon. \qquad (4.4)\end{aligned}$$

Again using (1.7) and (1.1) we have

$$\begin{aligned}\|x_{n+1} - \tilde{x}_{n+1}\| &= \| Ty_n - \tilde{T}\tilde{y}_n \| \\ &= \| Ty_n - T\tilde{y}_n + T\tilde{y}_n - \tilde{T}\tilde{y}_n \| \\ &\leq \|Ty_n - T\tilde{y}_n\| + \| T\tilde{y}_n - \tilde{T}\tilde{y}_n \| \\ &\leq \delta\|y_n - \tilde{y}_n\| + \varepsilon \\ &\leq \|y_n - \tilde{y}_n\| + \varepsilon. \qquad (4.5)\end{aligned}$$

Using (4.4) in (4.5), we obtain

$$\|x_{n+1} - \tilde{x}_{n+1}\| \le (1-(1-\delta)\alpha_n)\|x_n - \tilde{x}_n\| + 3\varepsilon + \varepsilon$$
$$\le (1-(1-\delta)\alpha_n)\|x_n - \check{x}_n\| + 4\varepsilon$$
$$\le \left(1-\alpha_n(1-\delta)\right)\|x_n - \tilde{x}_n\| + (1-\delta)\frac{4\varepsilon}{1-\delta}.$$

Let $\beta_n = \|x_n - \tilde{x}_n\|, \mu_n = (1-\delta) \in (0,1)$ and $\gamma_n = \dfrac{4\varepsilon}{1-\delta}$.

Using lemma 1.4, we have

$$0 \leqslant \limsup_{n\to\infty}\|x_n - \tilde{x}_n\| \le \lim_{n\to\infty}\frac{4\varepsilon}{1-\delta}.$$

From our earlier results, we have lim_(n→∞)x_n=p. Also, we have assumed that lim_(n→∞)x ̃_n=p ̃. we have ‖p-p ̃‖≤4ε/(1-δ). This completes the proof.

5. APPLICATION TO DELAY DIFFERENTIAL EQUATIONS

In this section, we have proved that F iteration scheme can be used to find the solution of delay differential equations. consider the space C[a,b] endowed with Chebyshev norm given by:

$$\| x - y \|_\infty = e^{ik} \max_{t\in[a,b]} |x(t) - y(t)|,$$
$$x, y \in c[a,b],\ k \in \left[0, \frac{\pi}{2}\right].$$

C[a,b] represents the set of all the continuous functions defined on the interval [a,b]. It is a Banach space. In the present work, we have studied the following delay differential equation:

$$x'(t) = f\left(t, x(t), x(t-\tau)\right), t \in [t_0, b], \qquad (5.1)$$

with initial condition

$$x(t) = \psi(t), t \in [t_0 - \tau, t_0]. \qquad (5.2)$$

Assuming that following conditions hold:

(i) $t_0, b \in R, \tau > 0$,

(ii) $f \in C([t_0, b] \times R^2, R]$,

(iii) $\psi \in C[t_0 - \tau, b], R)$,

(iv)$2L_f(b - t_i) < 1$.

Hence, there exists L_f>0 such that

$$|f(t, q_1, q_2) - f(t, r_1, r_2)| \le l_f \sum_{i=1}^{2} |q_i - r_i| \ \forall q_i, r_i \in R. \qquad (5.3)$$

By a solution of delay differential equation (5.1),(5.2), we mean a function ⊦ x∈C[t_0-τ,b],R∩C^' (t_0,b],R). First of all, we reformulate delay differential equation (5.1)- (5.2) in the form of following integral equation:

$$x(t) = \begin{cases} \psi(t), & t \in [t_0 - \tau, t_0], \\ \psi(t_0) + \int_{t_0}^{t} f(\beta, x(\rho), x(\beta - t))d\rho, & t \in [t_0, b]. \end{cases} \quad (5.4)$$

Now, we prove the main result of this section.

Theorem 5.1: Assume that conditions (i)- (v) are satisfied. The delay differential equations (5.1)- (5.2) has a unique solution say p in $C([t_0 - \tau, b], R) \cap C'([t_0, b], R)$ and F iteration process converges to p.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by F iterative scheme for the operator (5.4). Let p be the fixed point of T. We need to show that $x_n \to p$ as $n \to \infty$. Clearly, $x_n \to p$ for each $t \in [t_0 - \tau, t_0]$. We need only to establish the result for the interval $[t_0, b]$.

We have

$$\begin{aligned}
\|z_n - p\|_\infty &= \| T\big((1-\alpha_n)x_n + \alpha_n T x_n\big) - p \|_\infty \\
&\le e^{ik} \max_{t\in[t_0-\tau,b]} \| \big((1-\alpha_n)x_n + \alpha_n T x_n\big) - p \|_\infty \\
&\le e^{ik} \max_{t\in[t_0-\tau,b]} \left[(1-\alpha_n)\|x_n - p\|_\infty + \alpha_n \|T x_n - p\|_\infty\right] \\
&\le e^{ik} \max_{t\in[t_0-\tau,b]} [(1-\alpha_n)\|x_n - p\|_\infty + e^{ik}\alpha_n \max_{t\in[t_0-\tau,b]} | T_{x_n}(t) - T_{p(t)}|] \\
&+ \int_{t_0}^{t} f(s, x_n(s), m_n(s-\tau))dp - | \psi(t_0) - \int_{t_0}^{t} f(s, p(s), P(\ell - \tau))ds | \\
&= e^{ik} \max_{t\in[t_0-\tau,b]} \Big[(1-\alpha_n)\|x_n - p\|_\infty + \alpha_n e_{ik} \max_{t\in[t_0-\tau,b]} | \int_{t_0}^{t} f(s, x_n(s), \\
&x_n(s-\tau))dp - \int_{t_0}^{t} f(s, p(s), p(s-\tau))ds | \\
&\le e^{ik} \max_{t\in[t_0-\tau,b]} \int_{t\in[t_0-\tau,b]}^{t} L_f |z_n(s) - P(s)| + |z_n(\Delta - \tau) - p(s-\tau)| ds \\
&\le 2L_f(b - t_0)e^{ik}\|z_n - p\|_\infty.
\end{aligned}$$

Again using (1.7), we have

$$\|x_{n+1} - p\|_\infty \quad = \|T_n - p\|_\infty$$

$$\leq e^{ik} \max_{t\in[t_0-\tau,b]} | \int_{t_0}^{t} f(s, y_n(s), y_n(s-\tau)) - f(s, p(s), p(s-\tau))ds |$$

$$\leq e^{ik} \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} | f(s, y_n(s), y_n(s-\tau)) - f(s, p(s), p(s-\tau)) | ds$$

$$\leq e^{ik} \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f(| y_n(s) - p(s) + z_n(s-\tau) - p(s-\tau) | ds$$

$$\leq \ 2L_f(b-t_0)e^{ik}\|y_n - p\|_\infty.$$

Combining (5.6) and (5.7), we have

$$\|y_n - P\|_\infty \leq \ 2L_f(b-t_0)e^{ik} \cdot e^{ik} \max_{t\in[t_0-\tau,b]} \left[1 - \alpha_n \left(1 - 2L_f(b-t_0)\right) \|x_n - P\|_\infty\right.$$

$$\leq 2L_f(b-t_0)e^{2ik}\left(1 - \alpha_n\left(1 - 2L_f(b-t_0)\right) \max_{t\in[t_b-\tau,b]} \|x_n - p\|_\infty. \quad (5.9)$$

Combining (5.8) and (5.9)

$$\|x_{n+1} - p\|_\infty \leq 2L_f(b-t_0)c^{ik}[2l_f(b-t_0)e^{2ik}(1-\alpha_n(1-2L_f(b-l)$$
$$\max_{[t_1-\tau,b]} \|x_n - p\|_\infty$$
$$\leq \left(2l_f(b-t_0)\right)^2 e^{3ik}\left(1-\alpha_n\left(1-2L_f(b-t_0)\right)\right.$$
$$2L_f(b-t_0)e^{ik}\|x_n - p\|_\infty$$
$$\leq \left(2L_f(b-t_0)^3 c^{4ik}\left(1-\alpha_n\left(1-2L_f(b-t_0)\right)\|x_n - p\|_\infty. \quad (5.10)\right.\right.$$

Applying assumption (v) on (5.10), we have

$$\|x_{n+1} - p\|_\infty \leq e^{4ik}\left[1-\alpha_n\left(1-2l_f(b-t_0)\right)\right]\|x_n - p\|. \ (5\cdot 11)$$

Using induction on (5.I1), we have

$$\|x_{n+1} - p\|_\infty \leq e^{4ik}\prod_{i=1}^{n}\left[1-\alpha_i\left(1-2L_f(b-t_0)\right)\right]\|x_1 - p\|.$$

Since $\alpha_n \in [0,1]$ for $a \parallel n \in N$ and by using assumption (v) we obtain

$$\left[1-\alpha_n\left(1-2l_f(b-t_0)\right)\right] < 1.$$

Hence,

$\|x_{n+1} - p\|_\infty \leq \frac{|e^{4ik}\|x_1 - p\|_\infty|}{\left|e^{\left(1-2l_f(b-t_0)\right)\sum_{i=1}^{n}\alpha_i}\right|}$. Hence, $\lim_{n\to\infty}\|x_{n+1} - p\|_\infty \to 0$ as $n \to \infty$.

Hence,

It means that lim_(n→∞)‖x_n-p‖_∞=0. This complete the proof.

Remark 5.1. Theorem 5.1 generalizes various results available in literature including the work of Okeke [18], Common et. al. [20] and Okeke and Abbas [15].

Concluding Remarks: The present work reports some fixed point results based on contractive condition and contractive condition of rational expression. Numerically in the Table 2.1 and Table 2.2, it is claimed the F-iteration scheme is more efficient than some of the existing iterative schemes available in the literature. Even the change in the initial value does not reduce the efficiency of F-iteration scheme as claimed with the help of Table 2.3. As an application to our results we have proposed solution to a delay differential equation. Our results are improvement of many results including the work of Okeke [18], Common et. al. [20] and Okeke and Abbas [15]. Further, there are some work in the future like developing some more efficient iterative schemes than F-iteration, replacing or changing some conditions in our main results, extending our results to another metric spaces like fuzzy metric space, b- metric space etc.

CONFLICT OF INTERESTS

None.

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None.

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