Original Article
ISSN (Online): 2582-7472

ShodhKosh: Journal of Visual and Performing Arts
June 2024 5(6), 815–829

ANALYZING FUZZY MATRICES AND CONNECTIVITY IN SPECTRAL FUZZY GRAPH THEORY

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DOI
10.29121/shodhkosh.v5.i6.2024.4301

Funding: This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

ABSTRACT

In this paper, an analysis is conducted on several fuzzy matrices associated with a fuzzy graph, including the adjacency matrix A(G) and the Laplacian matrix L(G). The eigenvalues of the adjacency matrix (λi) of a fuzzy graph and their properties are stated and discussed. Fuzzy vertex connectivity (κ), along with algebraic connectivity ($\vartheta n-1$), adjacency, and Laplacian eigenvalues, are studied under conditions $\kappa \leq q$ to make contributions to spectral fuzzy graph theory, enhancing the connectivity in network structures due to the occurrence of linguistic and inexact variables. Additionally, the relation- ship between κ and $\vartheta n-1$ shows the strength of connectivity among the vertices. Keywords: Adjacency eigenvalue, laplacian eigenvalue, fuzzy vertex connectivity, second smallest laplacian eigenvalue, maximum strong degree, minimum strong degree.

Keywords: Adjacency Eigenvalue, Laplacian Eigenvalue, Fuzzy Vertex Connectivity, Second Smallest Laplacian Eigenvalue, Maximum Strong Degree, Minimum Strong Degree
AMS Subject Classification: 05C40, 05C50, 05C72, 58C40

1. INTRODUCTION

Spectral graph theory entails the analysis of the spectral properties of specific matrices derived from a given graph. These matrices encompass fundamental representations like the adjacency matrix and the Laplacian matrix, along with other related ones. The exploration of graph spectra has been a well- established field of research for more than half a century. Nevertheless, within the last fifteen years, there has been a noticeable surge in interest surrounding the investigation of generalized Laplacian ma- trices associated with graphs. For a comprehensive overview of the findings in spectral graph theory and the inverse eigenvalue problem of graphs, readers can refer to [11]. This dredges the intricate intercon- nections and introduces pioneering outcomes pertaining to the construction of matrices with minimal rank, specifically tailored for distinct graph types such as [0, 1]-matrices or generalized Laplacians. The fundamental terminology and the characterization of graphs with minimal rank in this context were established by Leslie Hogben in [3]. The second largest eigenvalue ($\lambda 2$) and second smallest Lapla-cian eigenvalue ($\vartheta n-1$) of a graph serve as critical indicators of its connectivity, as they are intimately connected to connectivity attributes such as vertex and edge connectivity.

In [1], the upper bounds for the second-largest eigenvalues of regular graphs and multigraphs with a specified order are presented. These bounds ensure a predefined level of vertex connectivity and are expressed in terms of the graph's order and degree. Additionally, in [17], analogous results concerning the properties $\vartheta n-1(G)$ and $\lambda 2(G)$ are discussed in the context of characterizing the vertex connectivity of regular graphs, triangle-free graphs, and graphs with a fixed girth.

Fuzzy models, as highlighted in [16], emerge as valuable tools for effectively addressing uncertainty and achieving solutions characterized by accuracy and precision. The foundation for this approach was laid by Zadeh in 1965 when he introduced fuzzy sets as a means of handling uncertainty in representa- tion. The concept of fuzzy graphs (FGs) was introduced by Kaufmann based on Zadeh's fuzzy relations. The inception of FGs dates back to 1975 when they were independently introduced by Rosenfeld and Yeh with Bang. A concise overview of fuzzy relations and fuzzy groups (1991) was provided in [12]. In the domain of FGs, Rosenfeld made seminal contributions by establishing fundamental concepts related to structure and connectivity [5]. On the other hand, Yeh and Bang's work extended the field by intro- ducing a variety of connectivity parameters specific to FGs. Their research also delved into the practical applications of these parameters, particularly in the domain of clustering analysis.

In [4], Bhattacharya further contributed to this field by introducing novel connectivity principles centered around fuzzy cut nodes and fuzzy bridges. Additionally, Bhattacharya introduced the concept of fuzzy groups and metric notions. Furthermore, researchers from various quarters have dedicated efforts to crafting algorithms aimed at determining node connectivity and degrees of connectedness within FGs. These endeavors have also led to the creation of algorithms for computing connectedness matrices in the context of FGs.

Fuzzy graph theory offers a broader framework compared to classical graph theory, albeit with dis- tinctive characteristics. In classical graphs, the measure of connectivity between vertices is binary, represented as either 0 or 1. However, in FGs as outlined in [16], this measure can assume any real value within the range [0, 1]. This real-valued notion of connectedness extends to paths and cycles within FGs. Furthermore, in [8], an extensive inquiry is conducted exploring the attributes associated with diverse degrees, orders, and sizes of FGs. This enhanced effectiveness stems from the capability of fuzzy graphs to adeptly manage the varying strengths of edges. The research conducted by Sunil Mathew and Sunitha, which focuses on connectivity concepts in FGs and their practical utilization in clustering and network analysis, not only underscores the current advancements but also suggests potential directions for future research. This includes further exploration of how connectivity concepts in FGs can be harnessed to address various challenges within network-related problem-solving.

The present study attempts to tackle shortcomings observed in existing parameters while striving to establish a more resilient framework for scrutinizing fuzzy graphs within real-world contexts. This research extends knowledge by focusing on the relationship between fuzzy vertex connectivity and al-

gebraic connectivity in FGs, as the connectivity bounds can be easily attained. Theoretical contributions on $\vartheta n-1(G)$ and κ yield the fact that $\vartheta n-1 \simeq \kappa$.

The paper is organized as follows: Section 2 provides the foundational concepts and background information. A comprehensive analysis of the eigenvalues of fuzzy adjacency matrices and fuzzy Lapla- cian matrices, along with theorems describing combinatorial properties, is established in Section 3. In Section 4, fuzzy vertex connectivity in fuzzy graphs, and its associated properties are explored. It is shown that the fuzzy vertex connectivity is less than or equal to a particular membership value (q) un- der specified conditions. Additionally, the maximum and minimum strong degrees in the context of fuzzy graphs, drawing connections to the second smallest Laplacian eigenvalues are examined. Section 5 concludes the paper.

2. PRELIMINARIES

A fuzzy graph $G = (V, \sigma, \mu)$ is defined by a set V where each vertex is associated with a mem- bership function σ mapping to [0, 1], and an edge set μ mapping pairs of vertices to [0, 1], satisfy- ing $\mu(xy) \leq \sigma(x) \wedge \sigma(y)$ for all vertices $x, y \in V$. The adjacency matrix of a fuzzy graph G is $A = [a_{ij}]$, where $a_{ij} = \mu(\sigma i, \sigma j)$ denotes the strength of connection between vertices σi and σj· The eigenvalues λi of A, ordered from largest to smallest, form the adjacency spectrum or fuzzy spectrum of G· The eigenvalues are denoted by $\lambda_i : \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$ of A· The Laplacian matrix L(G) = D(G) − A(G), where D(G) is a diagonal matrix representing vertex degrees, pro- vides Laplacian eigenvalues ϑ_i, ranging from 0 to the largest eigenvalue. These Laplacian eigenval- ues are symbolized as $0 = \vartheta_n \leq \vartheta_{n-1} \leq \ldots \leq \vartheta_2 \leq \vartheta_1$· The algebraic connectivity ϑ_{n-1} corre- sponds to the Fiedler eigenvalue, crucial for understanding graph connectivity patterns. Fuzzy distances $fd(\mu_i, \mu_j)$ between vertices are computed using minimum operations on membership values σ and con- nection strengths μ· Fuzzy vertex connectivity κ(G) is defined as the minimum strong weight [13] of a fuzzy vertex cut, where attributes like strong edge weights ($d_s(\sigma_i)$), minimum ($\delta_s(G)$), and maximum ($\Delta_s(G)$) vertex degrees [13], and overall edge strengths illustrate the robustness of G· A set of vertices $J = \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \subset V$ is said to be a fuzzy vertex cut if $CONN_{G-J}(\sigma_i, \sigma_j) < CONN_G(\sigma_i, \sigma_j)$.

The order $n*$ of G sums all vertex memberships, providing a measure of the graph's scale and com- plexity. A fuzzy graph G is complete [9] if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. A fuzzy graph G

is called a regular fuzzy graph if every vertex in the graph has the same degree whereas G is called a totally regular fuzzy graph [9] if every vertex has the same strong degree [13]. The strong degree $d_s(\sigma_i)$ of G associated with a vertex σ_i is determined by summing the membership values of all strong edges [14] incident at that vertex.

Example 2.1. Consider a fuzzy graph G with four vertices $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ with the vertex and edge membership as $\sigma_1 = 0.5$, $\sigma_2 = 0.3$, $\sigma_3 = 0.2$, $\sigma_4 = 1$ and $\mu_{12} = 0.2$, $\mu_{23} = 0.1$, $\mu_{34} = 0.1$, $\mu_{14} = 0.4$, $\mu_{24} = 0.3$·

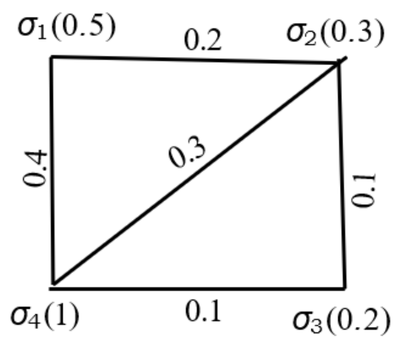

Figure 2.1 Fuzzy Graph G depicting vertex connectivity and fuzzy spectrum

The adjacency matrix of a fuzzy graph G is given by, $A = \begin{bmatrix} 0 & 0.2 & 0 & 0.4 \\ 0.2 & 0 & 0.1 & 0.3 \\ 0 & 0.1 & 0 & 0.1 \\ 0.4 & 0.3 & 0.1 & 0 \end{bmatrix}$

The fuzzy spectrum of G yields values $\lambda(G) = \{0.6284, 0.0071, -0.2104, -0.4252\}$. The Laplacian matrix L(G) = D(G)− A(G), provides Laplacian eigenvalues $\vartheta(G) = \{0, 0.2528, 0.8, 1.1472\}$.

Also, $\Delta_s = 0.8$, $\delta_s = 0.2$, $n* = 2$· The fuzzy vertex cut (FVC) sets of Fig.2.1 are denoted such that $J_1 = \{\sigma_3\}$ is the only 1-FVC with $s(J_1) = 0.1$. The only 2-FVC in G is $J_2 = \{\sigma_2, \sigma_4\}$ and $s(J_2) = 0.2$. Any three vertices of G form a 3-FVC with $s(\{\sigma_1, \sigma_3, \sigma_4\}) = s(\{\sigma_1, \sigma_2, \sigma_3\}) = s(\{\sigma_1, \sigma_2, \sigma_4\}) = 0.6$ and $s(\{\sigma_2, \sigma_3, \sigma_4\}) = 0.3$. Thus, κ = 0.1·

3. PROPERTIES ON SPECTRUM OF FGS

Fiedler introduced algebraic connectivity [7], vital for understanding graph robustness. Brouwer and Haemers (2011) detailed the spectral properties of graphs [6], while Jiang offerred an accessible introduction to spectral graph theory [10]. In this section, the theorems presented and proven articulate the spectral properties of simple connected, and regular fuzzy graphs with n vertices.

The theorem below relates the eigenvalues of the subgraph induced by the cut vertex and arranged in a non-increasing order.

Theorem 3.1. Let λi, i = 1, 2, . . . , n be the spectrum of a fuzzy graph G. If a cut vertex of a fuzzy graph induces a subgraph, then the spectrum of the resultant fuzzy graph GA are λ∗, i = 1, 2, . . . , n − 1, then

$$\lambda_1^* \geq \lambda_2^* \geq \ldots \geq \lambda_{n-2}^* \geq \lambda_{n-1}^*$$

Proof. Let G = (σ, µ) be a fuzzy graph with n vertices and m edges. The adjacency matrix A of a fuzzy graph G is a matrix where each entry Aij represents the weight (or membership value) of the edge between vertices i and j. The eigenvalues of the adjacency matrix A are λ1 ≥ λ2 ≥ . . . ≥ λn.

Removing a cut vertex v from G results in a subgraph GA with adjacency matrix AA, which is a prin- cipal submatrix of A. Let λ∗ be the eigenvalues of AA. By the properties of principal submatrices, the eigenvalues of AA interlace with the eigenvalues of A.

The Interlacing Theorem states that if AA is a principal submatrix of A, then the eigenvalues of AA

interlace with those of A.

$$\lambda_1 \geq \lambda^*_1 \geq \lambda_2 \geq \lambda^*_2 \geq \ldots \geq \lambda^*_{n-1} \geq \lambda_n$$

This means that each λ_i* lies between λ_i and λ_i+1. Since the eigenvalues λ_i* of A_A interlace with those of A, the ordering of $\lambda*_i$ is preserved in a non-increasing manner. Specifically, for AA, the eigenvalues $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_{n-1}^*$ are bounded by λ_i and λ_{i+1} of A ensuring

$$\lambda_1^* \geq \lambda_2^* \geq \ldots \geq \lambda_{n-2}^* \geq \lambda_{n-1}^*.$$

The eigenvalues of AA (the adjacency matrix of the subgraph) are not only interlaced but also ordered non-increasingly due to the inherent properties of the adjacency matrix and the interlacing theorem.

Thus the removal of a cut vertex results in a subgraph whose spectrum interlace with those of the original graph, preserving their non-increasing order.

Corollary. Let G = (σ, µ) be a fuzzy graph and suppose Vc is a cut vertex of G. If the largest component of G − $\{V_c\}$ contains r vertices, then

$$r - 1 \geq \lambda_2(G).$$

Proof. It is known that $\lambda_1(G - \{V_c\}) \leq r$. By theorem 3.1, there exists $\lambda_2(G) \leq r - 1$.

The following example illustrates this concept.

Example 3.2. Consider the subgraph GA as given below of graph G as depicted in Fig.2.1.

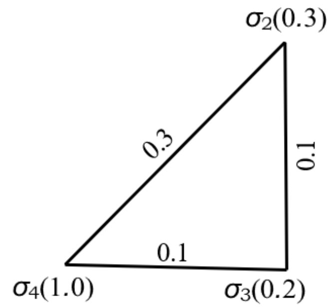

Figure 3.1 Fuzzy Subgraph GA obtained from Graph G

The subgraph GA has eigenvalues $\lambda* = \{0.4113, -0.0911, -0.3202\}$ that satisfies $\lambda* \geq \lambda2 \geq \ldots \geq \lambda* \quad \geq \lambda n-1$.

Following theorems depicts that, if the graph is totally regular, the largest eigenvalue equals the degree. Otherwise, the largest eigenvalue is between the average degree and the maximum strong degree, and also between the minimum strong degree and the square root of the maximum strong degree.

Theorem 3.3. Let G be a fuzzy graph with largest spectrum $\lambda1$. If G is a totally regular fuzzy graph of degree d, then $\lambda1 = d$. Otherwise, $\delta_s < d_g < \lambda_1 < \Delta_s$ where δ_s, Δ_s, dg denotes minimum strong, maximum strong, and average degrees of fuzzy graphs.

Proof. Assume G is a totally regular fuzzy graph with degree d. This means every vertex in G has the same degree d. Let λn, $(n = 1, 2, \ldots)$ be the sum of absolute eigenvalues with adjacency matrix A. Given that, $\lambda1$ has the largest spectrum for any G that is regular, then $\lambda1 = d$.

By Perron-Frobenius theorem [6], for a non-negative, irreducible matrix A, the largest eigenvalue (spec- tral radius) is real and positive. For each eigenvalue λ of G, we have $|\lambda| \leq \lambda1$. If G is primitive, then $|\lambda| = \lambda_1 \Longrightarrow \lambda = \lambda_1$.

For a totally regular fuzzy graph, the eigenvector corresponding to $\lambda1 = d$ is a constant vector. Since G is totally regular, $\lambda1 = d$.

For a non-regular fuzzy graph, the degrees of vertices are not all equal. Since G is connected and undi- rected, the largest eigenvalue $\lambda1$ of the adjacency matrix A satisfies: $\delta s \leq \lambda1 \leq \Delta s$.

By the properties of the average degree, $\delta s \leq dg \leq \Delta s$. Because $\lambda1$ is the largest eigenvalue, it will be greater than the average degree dg but less than the maximum strong degree Δ_s. Since G is connected, the minimum strong degree δs will be less than the average degree dg. Combining these inequalities, we get $\delta s < dg < \lambda_1 < \Delta_s$.

Theorem 3.4. Let G be any fuzzy graph with n vertices. Then,

1) $d_g \leq \lambda_1 \leq \Delta_s$

2) $\delta_s \leq \lambda_1 \leq \sqrt{\Delta s}$

Proof. Let $G = (\sigma, \mu)$ be a fuzzy graph with n vertices. For any G with /IV /I = n where V is a vertex set and $E = \{\mu_1, \mu_2, \ldots, \mu_m\}$ denotes the edge set having the eigenvalues of A with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. We know that the adjacency matrix of the fuzzy graph G is a non-negative, symmetric matrix. By the Perron-Frobenius theorem for fuzzy matrices,

- There exists a unique non-negative eigenvector v corresponding to the largest eigenvalue λ_1.
- All other eigenvalues of the adjacency matrix have modulus strictly less than λ_1.

Since v is non-negative and $v^T v = 1$, it follows that $\lambda1$ is greater than or equal to the minimum degree d_G of vertices in G. Hence, $d_G \leq \lambda_1$. Similarly, since all entries of the adjacency matrix are atmost 1, it follows that $\lambda1$ is less than or equal to the maximum degree Δs of vertices in G. Hence, $\lambda1 \leq \Delta s$.

Therefore, the inequality $d_g \leq \lambda_1 \leq \Delta_s$ holds.

Similarly, we can establish that $\delta s \leq \lambda_1 \leq \sqrt{\Delta_s}$. Since the strength of vertices in G are analogous to the degrees of vertices in a crisp graph, the proof follows similar lines as in property i), applying the Perron-Frobenius theorem for fuzzy matrices.

Theorem 3.5. For any fuzzy graph G on n vertices, the following conditions hold:

- $\sum_i \lambda_i \leq$ with equality holding iff G has no isolated vertices and loops
- For $n \geq 2$, $\lambda_1 \leq {}_{n-1}$ with equality holding iff G is a complete fuzzy graph on n vertices
- For a fuzzy graph G which is not complete implies $\lambda i \leq 1$

Proof. Let G be a fuzzy graph with n vertices. $\sum_i \lambda_i$ follows from considering the trace of a Laplacian fuzzy graph L(G). Assuming i = 1, as λ_1 yields the largest eigenvalue of G, $\lambda_1 \leq n$.

Also, the sum of absolute eigenvalues λi of A is zero. The inequality for n $\geq 2,\ \lambda_1 \leq \frac{n}{n-1}$ follows from S $\sum_i \lambda_i \leq n$ that,

If G has no isolated vertices and loops, $\sum_i \lambda_i \leq n.$ For n = 3, the fuzzy graph holds the eigenvalue lesser than $\frac{n}{n-1}$. Suppose G has two non-adjacent vertices x and y. Let W be the diagonal matrix (v, v)th entry value dv defined by,

$$W = \begin{cases} d_y & \text{if } v = x \\ -d_a & \text{if } v = y \\ 0 & \text{if } v \neq x, y \end{cases}$$

Then

$$\lambda_G = \lambda_1 = \frac{inf_{f \perp W} \sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v}$$

The dirichlet sum of overall unordered pairs {u, v} to the sum of the squares of f (v) equals the largest eigenvalue in G. Therefore, for any non-complete fuzzy graph G, $\lambda_1 \leq {}_1$.

The relationship between the determinant, Laplacian and adjacency eigenvalues are discussed here.

Theorem 3.6. Let G be a regular fuzzy graph with n vertices. Suppose that the eigenvalues of L(G) are $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \vartheta_n$ with $\vartheta_n = 0$. Then

$$det(L_0) < \frac{1}{n}[\vartheta_1 \cdot \vartheta_2 \ldots \vartheta_{n-1}]$$

where L0 is the fuzzy laplacian matrix with ith row and ith column deleted for all i = 1, 2, . . . , n.

Proof. G is a regular fuzzy graph whenever there exists a walk with membership degree between any two vertices. For any Laplacian matrix L(G), Let L_0 be a matrix obtained by deleting the row and column i in G. Let $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \vartheta_n$ be the eigenvalues of L(G) with $\vartheta_n = 0$. The determinant of L_0 is related to the determinant of L(G) as follows:

$$det(L_0) = det(L(G)) \times (\vartheta_i)^{-1} \text{ for any } i = 1, 2, \ldots, n-1.$$

We know that $\vartheta_n = 0$ is an eigenvalue of L(G), implying that det(L(G)) = 0. Removing the i^{th} row and i^{th} column of L(G) to obtain L_0 affects the fact that its determinant is non-zero. Therefore, $det(L_0) \neq 0.$. since, $det(L_0) \neq 0$ the given expression $\frac{1}{n}[\vartheta_1.\vartheta_2 \ldots \vartheta_{n-1}]$ is also non-zero. Therefore,

$$det(L_0) < \frac{1}{n}[\vartheta_1 \cdot \vartheta_2 \ldots \vartheta_{n-1}]$$

Theorem 3.7. Suppose that G is a regular fuzzy graph of degree ds and the eigenvalues of A(G) are $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n$ *with* $\lambda_n = d_s$. *Then,* . Then,

$$det(L_0) \neq \frac{1}{n}[(d-\lambda_1) \cdot (d-\lambda_2) \ldots (d-\lambda_{n-1})]$$

where L_0 is the fuzzy laplacian matrix with i^{th} row and i^{th} column deleted for all i = 1, 2, . . . , n.

Proof. Let G be a regular fuzzy graph of degree d_s with n vertices. Let A(G) be the adjacency matrix obtained from G with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$, λn with $\lambda_n = d_s$. L_0 denotes the Laplacian matrix obtained by deleting the i^{th} row and i^{ith} column for i = 1, 2, . . . , n. To prove the statement, utilize the fact that the determinant of a matrix changes upon deletion of a row and a column from it. The determinant of L0 is related to the determinant of L(G) as follows:

$$det(L_0) = det(L(G)) \times (d-\lambda_i)^{-1} \text{ for any } i = 1, 2, \ldots, n-1$$

Let us prove this theorem by contradiction. Assume that the given statement holds true, i.e.,

$$det(L_0) = \frac{1}{n}[(d-\lambda_1).(d-\lambda_2) \ldots (d-\lambda_{n-1})]. \tag{3.1}$$

From the relationship between determinants, we see that

$$det(L(G)) \times (d-\lambda_i)^{-1} = \frac{1}{n}[(d-\lambda_1).(d-\lambda_2) \ldots (d-\lambda_{n-1})] \text{ for any } i = 1, 2, \ldots, n-1.$$

Multiplying both sides by n, we get

$$n \times det(L(G)) = [(d-\lambda_1).(d-\lambda_2) \ldots (d-\lambda_{n-1})] \times \Pi_{i=1}^{n-1}(d-\lambda_i)^{-1}$$

Simplifying, we have:

$$n \times det(L(G)) = [(d-\lambda_1) \cdot (d-\lambda_2) \ldots (d-\lambda_{n-1})] \tag{3.2}$$

Since det(L(G)) is a constant, this would imply that the right-hand side is also a constant. However, this contradicts the fact that the eigenvalues λi can vary, and thus the right-hand side is not constant.

Therefore, our assumption that the statement holds true leads to a contradiction. Therefore, $\frac{1}{n}[(d-\lambda_1)\cdot(d-\lambda_2)\ldots(d-\lambda_{n-1})]$ not equals the determinant of the reduced Laplacian matrix provided G is regular.

The following example illustrates this concept.

Example 3.8. Consider the regular fuzzy graph G as depicted in Fig.3.2.

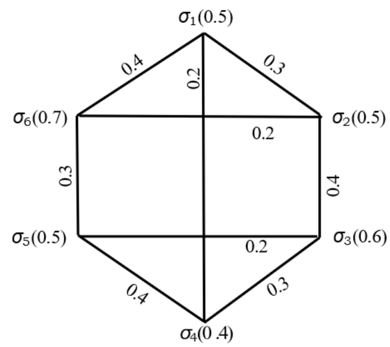

Fig.3.2 Regular fuzzy graph G with det(L0) = 0.17 and λ_s = ds = 0.9

Here, det(L_0) = 0.17 and ϑ_i = {1.573, 1.4558, 1.0732, 0.7268, 0.5712, 0} ; λi = {0.9, 0.3288, 0.1732, −0.1732, −0.5558, −0.6730} with $\lambda_s = d_s = 0.9$. This makes clear that

$$\frac{1}{n}[(d-\lambda_1)\cdot(d-\lambda_2)\ldots(d-\lambda_{n-1})] \,!= det(L_0) < \frac{1}{n}[\vartheta_1.\vartheta_2\ldots\vartheta_{n-1}].$$

Next theorem guarantees a unique non-negative eigenvector for the largest eigenvalue of symmetric fuzzy matrices for connected FGs.

Theorem 3.9. Let G be a fuzzy graph. For any irreducible, aperiodic, and symmetric fuzzy matrix A, there exists a unique non-negative eigenvector v1 corresponding to its largest eigenvalue λ1. Further- more, all other eigenvalues of A have moduli strictly less than λ_1.

Proof. Let G be a fuzzy graph. Let A = [a_{ij}] be the symmetric fuzzy matrix representing the fuzzy graph. First, Let us show the existence of a non-negative eigenvector v1 corresponding to the largest eigenvalue λ_1.

Since A is symmetric, it has real eigenvalues. By the properties of fuzzy matrices, A is non-negative, irreducible, and aperiodic. By the Perron-Frobenius theorem for non-negative matrices [6], there exists atleast one non-negative eigenvector v1 corresponding to the largest eigenvalue λ_1.

Part i]: Let us prove the uniqueness of eigenvector.

Assume there are two non-negative eigenvectors v_1 and v_2 corresponding to the largest eigenvalue λ_1. Because A is symmetric, v_1 is orthogonal to v_2, as distinct eigenvectors of a symmetric matrix are orthogonal. Now, Let $x = v_1 + v_2$. Since v_1 and v_2 are both non-negative and orthogonal, x is also non-negative and $Ax = Av_1 + Av_2 = \lambda_1 v_1 + \lambda_1 v_2 = \lambda_1 x$.

By the Perron-Frobenius theorem for non-negative matrices, x must be a positive scalar multiple of v_1. Thus, $v_2 = \alpha v_1$, where α is a positive scalar. As v_2 is non-negative, α must be positive. Therefore, $v_1 = v_2$. Hence, the eigenvector v1 corresponding to λ_1 is unique.

Part ii]: All other eigenvalues have moduli strictly less than λ_1.

Let λ2 be another eigenvalue of A. Consider the corresponding eigenvector w. Because A is symmetric, choose w to be orthogonal to v_1. Now, consider $v_1^T Aw.$ Since w is orthogonal to $v_1,\ v_1^T Aw = w^T Av_1$. But A is symmetric, so $w^T Av_1 = \lambda_1 w^T v_1.$.

If λ_2 have larger magnitude than λ_1, then $v_1^T Aw = \lambda_1 w^T v_1$ would be strictly greater than $\lambda_2 w^T v_1$, which contradicts the orthogonality of v1 and w. Thus, all other eigenvalues of A have moduli strictly less than λ_1. Therefore, λ_1 dominates over all other eigenvalues.

4. FUZZY VERTEX CONNECTIVITY

Vertex connectivity in fuzzy graphs extends classical connectivity concepts by incorporating vertex and edge membership values. Key contributions from Kaufmann and Gupta (1991), Rosenfeld [5], and Akram and Dudek (2010), Mathew and Sunitha [13], focuses on robust algorithms and applications shaping theoretical advancements and practical applications. In this section, the relationship between fuzzy vertex connectivity κ and second smallest Laplacian eigenvalue ϑ_{n-1} is established and it is proved that κ ≤ q under specified conditions on $\xi_1(\Delta_s, \delta_s, q)$ and $\xi_2(\Delta_s, \delta_s, q)$. Also, the upper bounds for $\frac{\vartheta_1}{\vartheta_{n-1}}$ and lower bounds for ϑ_{n-1} are presented to show $\vartheta_{n-1} \simeq \kappa$.

Theorem 4.1. Let q be any real number with 0 < q ≤ 1 and G denote a fuzzy graph of order n∗ with δ_s as minimum strong degree. If $\lambda_2(G) \le |\delta_s - q|$, then $\vartheta n-1(G) < q$.

Proof. Let A, L and D be the adjacency matrix, Laplacian matrix and diagonal matrix of G respectively. Since L = D−A, $\lambda_n(D)-\lambda_2(A) \le \lambda_{n-1}(L) = \delta_s$. Hence, $\vartheta_{n-1}(G) \ge \delta_s-\lambda_2(G)$. If $\lambda_2(G) \le |\delta_s-q|$ for any q ∈ (0, 1] (consider q = 0.5), $\vartheta n-1(G) \ge \delta_s - \lambda_2(G) < q$. Therefore, for any fuzzy graph G with n∗ and δ_s, $\vartheta n-_1 < q$.

Conditions on $\xi_1(\Delta_s, \delta_s, q)$ and $\xi_2(\Delta_s, \delta_s, q)$ are defined to find the lower bounds and upper bounds of fuzzy graphs and regular fuzzy graphs as below.

Definition 4.1. For any q ∈ (0, 1], n∗, Δs, δ_s, ds with $\delta_s \le q$ and $d_s \ge q$ define,

$$\text{(i)}\ \ \xi_1(\Delta_s, \delta_s, q) = \frac{(1-q)(n^*-q+1)\Delta_s}{4(\delta_s-q+2)(n^*-\delta_s-1)}$$

$$\text{(ii)}\ \ \xi_2(\Delta_s, \delta_s, q) = \frac{(1-q)(n^*)\Delta_s}{(n^*-q+1)(1-q)+4(\delta_s-q+2)(n^*-\delta_s-1)}$$

The following lemmas (Lemma 4.2, 4.3, 4.4) helps in deriving the relationship between fuzzy vertex connectivity κ and second smallest Laplacian eigenvalue ϑ_{n-1}.

Lemma 4.2. Let G = (V, σ, μ) be a fuzzy graph of order n∗. Let M, N be two disjoint fuzzy subsets of σ such that each vertex of M has a fuzzy distance atleast η to each vertex of N. Let μ_M, μ_N be the set of edges in G in M and N respectively. Then,

$$\vartheta_{n-1}(G) \le \frac{1}{\eta^2}\left(\frac{1}{|M|}+\frac{1}{|N|}\right)(|\mu|-|\mu_M|-|\mu_N|)$$

Lemma 4.3. Let G represent a fuzzy graph of order n∗. For all α ∈ Rn we have,

$$\vartheta_{n-1} = \bigwedge \frac{n^* \sum_{\mu_{ij}\in\mu}(\alpha_{\mu_i} - \alpha_{\mu_j})^2}{\sum_{\mu_i,\mu_j\in\sigma:\mu_i<\mu_j}(\alpha_{\mu_i} - \alpha_{\mu_j})^2}$$

Proof. Since $\vartheta_n(G) = 0$ and $\vartheta_n(G)$ is positive semidefinite, zero is the smallest Laplacian eigenvalue of G. Since G is connected, $(\vartheta_n, \mu_i) = \sum_{\mu_{ij}\in\mu} n^* \left(\alpha_{\mu_i} - \alpha_{\mu_j}\right)^2$. This implies that zero is a simple eigen- value and eigenvalues $\{\vartheta_1, \vartheta_2, ..., \vartheta_{n-1}\}$ are positive and the corresponding eigenvectors are orthogonal to ϑ_n. Thus $\vartheta_{n-1}(G)$ satisfies,

^ Lμij ∈μ n∗ (αμ − αμ)2

$$\vartheta_{n-1} = \bigwedge_{\mu\neq 0} \frac{\sum_{\mu_{ij}\in\mu} n^* \left(\alpha_{\mu_i} - \alpha_{\mu_j}\right)^2}{\sum_{\mu_i} \alpha_{\mu_i}^2}$$

and the minimum is attained for the eigenvectors corresponding to $\vartheta n(G)$. By the Lagrangian identity,

$$n^* \sum_{\mu_i} \alpha_{\mu_i}^2 - \left(\sum_{\mu_i} \alpha_{\mu_i}\right)^2 = \sum_{(\mu_i,\mu_j)} \left(\alpha_{\mu_i} - \alpha_{\mu_j}\right)^2$$

Lemma 4.4. Let G be a fuzzy graph of order $n*$ with minimum strong degree δ_s. Let R be the minimum fuzzy vertex-cut with κ vertices and M denote the set of all vertices having minimal components of G−R and N = V − (R ∪ M). Then,

$$|M|.|N| \geq (\delta_s - \kappa + 1)(n^* - \delta_s - 1)$$

Proof. Given that each vertex in the set M is connected to a maximum of |M | − 1 vertices within M and κ vertices within the set R,

$$\delta_s|M| \leq \sum_{\alpha\in M} d_s(\alpha) \leq |M|(|M| + \kappa - 1),$$

This implies that the cardinality of set M, denoted as |M |, must satisfy the inequality $|M| \geq \delta_s - \kappa + 1$. It is important to note that |M | is also bounded by |N | and together, they satisfy the equation $|M| + |N| = n* - \kappa$. Thus, $\delta s - \kappa + 1 \leq |M| \leq |N| \leq n* - \delta s - 1$.

The following theorems relate to the order of the fuzzy graph, the constraints on the strong degrees, the specific eigenvalue conditions, and the implications for the connectivity of the fuzzy graph.

Theorem 4.5. Let G be a fuzzy graph of order $n*$ with maximum strong degree Δ_s, minimum strong degree δ_s, strong degree $d_s(v)$ with $\Delta_s \geq q$, $_{n*} \geq 2_q$ and $d_s \geq q$. If $\vartheta n-1(G) > \xi_1(\Delta_s, \delta_s, q)$, then $\kappa(G) \leq q$.

Proof. Let us prove this theorem by contradiction. Assume that $1 - q \leq \kappa(G) \leq 1$. Let R be the mini- mum fuzzy vertex-cut with κ vertices and M denote the set of all vertices having minimal components of G − R and N = V − (R ∪ M). By Lemma 4.4, it follows

$$\delta_s - \kappa + 1 \leq |M| \leq |N| \leq n^* - \delta_s - 1 \tag{4.1}$$

$$|M|.|N| \geq (\delta_s - \kappa + 1)(n^* - \delta_s - 1) \tag{4.2}$$

$$|M|.|N| \geq (\delta_s - \kappa + 2)(n^* - \delta_s - 1) \tag{4.3}$$

$$|M|.|N| \geq (\delta_s - \kappa + 1)(n^* - \delta_s - 1) \geq (\delta_s - \kappa + 2)(n^* - \delta_s - 1) \tag{4.4}$$

Every edge in $\mu - (\mu_M \cup \mu_N)$ is adjacent to atleast $n* - |M| - |N|$ vertices in R. Thus,

$$|\mu| - |\mu_M| - |\mu_N| \leq (n^* - |M| - |N|)\Delta_s = \kappa\Delta_s \tag{4.5}$$

As $1 - q \leq \kappa \leq 1 \leq \frac{n^*}{2}$,(consider $q = 0.5$) we have

$$(n^* - \kappa)\kappa \leq (1 - q)(n^* - q + 1) \tag{4.6}$$

From Lemma 4.2,

$$\vartheta_{n-1}(G) \leq \frac{|M| + |N|}{4|M||N|}(|\mu| - |\mu_M| - |\mu_N|) \tag{4.7}$$

Substituting equations (4.4) and (4.5) in (4.7) by (4.6),

$$\begin{aligned}\vartheta_{n-1}(G) &\leq \frac{(|M| + |N|)\kappa\Delta_s}{4(\delta_s - q + 2)(n^* - \delta_s - 1)} \\ &= \frac{(n^* - \kappa)\kappa\Delta_s}{4(\delta_s - q + 2)(n^* - \delta_s - 1)} \\ &\leq \frac{(1 - q)(n^* - q + 1)\Delta_s}{4(\delta_s - q + 2)(n^* - \delta_s - 1)}\end{aligned}$$

This is a contradiction and $\kappa(G) \leq q$.

Theorem 4.6. Let G be a fuzzy graph of order $n*$ with maximum strong degree Δ_s, minimum strong degree δ_s, strong degree $ds(v)$ with $\delta_s \geq q$ and $d_s \geq q$. If $\vartheta_{n-1}(G) > \xi_2(\Delta_s, \delta_s, q)$, then $\kappa(G) \leq q$.

Proof. Let us prove this theorem by contradiction. Assume that $1 \leq \kappa(G) \leq 1 - q$. Let R be the minimum fuzzy vertex-cut with κ and M denote the vertex set having minimal components of $G - R$ and $N = V - (R \cup M)$. By Lemma 4.4,

$$\delta_s - \kappa + 1 \leq |M| \leq |N| \leq n^* - \delta_s - 1 \tag{4.8}$$

$$|M|.|N| \geq (\delta_s - \kappa + 1)(n^* - \delta_s - 1) \tag{4.9}$$

For any real vector $\alpha_i : i \in [0,1]$ and if $i \in M$, say $\alpha_i = 1$; if $i \in N$, $\alpha_i = 0.5$; $i \in R$, $\alpha_i = 0$. Denote $\mu_S = \mu \backslash (\mu_M \cup \mu_N)$). From Lemma (4.3), the equation holds for every α and applying to entries of α, it follows that

$$\sum_{\mu_{ij} \in \mu} (\alpha_{\mu_i} - \alpha_{\mu_j})^2 = \sum_{\mu_{ij} \in \mu_R} (\alpha_{\mu_i} - \alpha_{\mu_j})^2 \leq \sum_{\mu_{ij} \in \mu_R} 1 \leq |R|\Delta_s \leq (1-q)\Delta_s$$

$$\begin{aligned}\sum_{\mu_i,\mu_j \in \sigma : \mu_i < \mu_j} (\alpha_{\mu_i} - \alpha_{\mu_j})^2 &= \sum_{i \in M,\, \mu_j \in R} (\alpha_{\mu_i} - \alpha_{\mu_j})^2 + \sum_{i \in N,\, \mu_j \in R} (\alpha_{\mu_i} - \alpha_{\mu_j})^2 + \sum_{i \in M,\, \mu_j \in N} (\alpha_{\mu_i} - \alpha_{\mu_j})^2 \\ &= |M||R| + |N||R| + 4|M||N| \\ &= (n^* - \kappa)\kappa + 4|M||N| \\ &\leq (n^* - \kappa)\kappa + 4(\delta_s - \kappa + 1)(n^* - \delta_s - 1)\end{aligned}$$

This implies,

$$\vartheta_{n-1} \leq \frac{n^* \sum_{\mu_{ij} \in \mu} (\alpha_{\mu_i} - \alpha_{\mu_j})^2}{\sum_{\mu_i,\mu_j \in \sigma : \mu_i < \mu_j} (\alpha_{\mu_i} - \alpha_{\mu_j})^2} \leq \frac{(1-q)n^* \Delta_s}{(n^* - q + 1)(1-q) + 4(\delta_s - q + 2)(n^* - \delta_s - 1)}$$

which is a contradiction to our assumption and so, κ ≤ q.

Theorem 4.7. Let G be a fuzzy graph of order n∗ with minimum strong degree $0 \leq q \leq \delta_s$. *If* $\frac{\vartheta_1}{\vartheta_{n-1}} <$

$$r + \sqrt{r^2 - 1}, \textit{ then } \kappa(G) \leq q, \textit{ where}$$
$$r = \frac{4(\delta_s - q + 2)(n^* - \delta_s - 1)}{n^*(q+2)} + 1.$$

Proof. Let us prove this theorem by contradiction. Assume that 1 − q ≤ κ(G) ≤ 1. Let R be the mini- mum fuzzy vertex-cut with κ vertices and M denote the set of all vertices having minimal components of G − R and N = V − (R ∪ M). By Lemma 4.4,

$$\delta_s - \kappa + 1 \leq |M| \leq |N| \leq n^* - \delta_s - 1 \tag{4.10}$$

$$|M|.|N| \geq (\delta_s - \kappa + 1)(n^* - \delta_s - 1) \tag{4.11}$$

$$|M|.|N| \geq (\delta_s - \kappa + 2)(n^* - \delta_s - 1) \tag{4.12}$$

$$|M|.|N| \geq (\delta_s - \kappa + 1)(n^* - \delta_s - 1) \geq (\delta_s - \kappa + 2)(n^* - \delta_s - 1) \tag{4.13}$$

Combining these results with $n* - |M| - |N| = \kappa \leq 1 - q$, (consider $q = 0.5$) it implies that $\frac{|M||N|}{n^*(n^*-|M|-|N|)} \leq \frac{(\vartheta_1-\vartheta_{n-1})^2}{4\vartheta_1.\vartheta_{n-1}}$ if there is no such edge between M and N in G.

Therefore,

$$\frac{(\vartheta_1-\vartheta_{n-1})^2}{4\vartheta_1.\vartheta_{n-1}} \geq \frac{|M||N|}{n^*(n^*-|M|-|N|)} \geq \frac{(\delta_s-q+2)(n^*-\delta_s-1)}{n^*(q-1)} \tag{4.14}$$

Setting $h = \frac{\vartheta_1}{\vartheta_{n-1}}$ and $r = \frac{4(\delta_s-q+2)(n^*-\delta_s-1)}{n^*(q+2)} + 1$. Thus, equation (4.14) results in $h + h^{-1} \geq 2r$ and $h \geq 1,\ r \geq 1 \Longrightarrow h \geq r + \sqrt{r^2-1}$ which is a contradiction. Thus, $\kappa(G) \leq q$. □

Example 4.8. *Let G be a fuzzy graph with vertices $v_1, v_2, v_3, v_4, v_5, v_6$ with $\sigma_1 = 0.5, \sigma_2 = 0.3, \sigma_3 = 0.7, \sigma_4 = 0.4, \sigma_5 = 0.6$ and $\mu_{12} = 0.2, \mu_{23} = 0.2, \mu_{34} = 0.3, \mu_{45} = 0.1, \mu_{51} = 0.4, \mu_{25} = 0.1$.*

Here, $\Delta_s = 0.6,\ \delta_s = 0.3,\ \kappa = 0.2,\ n^* = O(G) = 2.5$. The fuzzy vertex cut (FVC) sets of Fig.4.1 are denoted such that $J_1 = \{v_2\}$ is the only 1-FVC with $s(J_1) = 0.2$. The only 2-FVC in G is $J_2 = \{v_2, v_5\}$ and $s(J_2) = 0.6$. Thus, $\kappa = 0.2$.

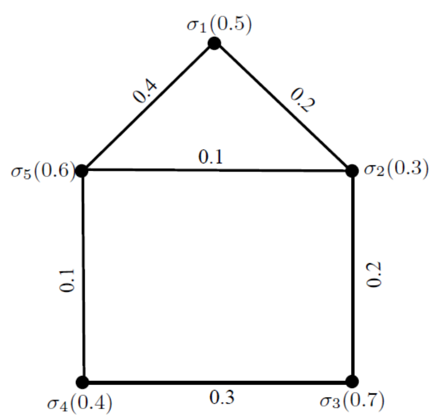

Fig.4.1 Fuzzy graph G to show the bounds with $\kappa = \vartheta_{n-1} = 0.2$

Table 4.1 : Fuzzy graph G to show the bounds on $\vartheta_{n-1}(G)$ and $\frac{\vartheta_1}{\vartheta_{n-1}}$

F-graph	$\vartheta_{n-1}(H)$	$\xi_1(\Delta_s, \delta_s, q)$	$\xi_2(\Delta_s, \delta_s, q)$	$\kappa(H)$	$\frac{\vartheta_1}{\vartheta_{n-1}}$	$r + \sqrt{r^2 - 1}$
G	0.2132	0.1041	0.0739	0.2	4.2767	4.5447

Table 4.1 shows the lower bounds on $\vartheta_{n-1}(G)$ of Theorems 4.5, 4.6 and the upper bound on $\frac{\vartheta_1}{\vartheta_{n-1}}$ of Theorem 4.7.

Example 4.9. *Let H be a regular fuzzy graph with vertices $v_1, v_2, v_3, v_4, v_5, v_6$ with $\sigma_1 = 0.5, \sigma_2 = 0.5, \sigma_3 = 0.6, \sigma_4 = 0.4, \sigma_5 = 0.5, \sigma_6 = 0.7$ and $\mu_{12} = 0.3, \mu_{23} = 0.4, \mu_{34} = 0.3, \mu_{45} = 0.4, \mu_{56} = 0.3, \mu_{61} = 0.4, \mu_{14} = \mu_{62} = \mu_{53} = 0.2$.*

Here, $\Delta_s = \delta_s = d_s = 0.7$, $\kappa = 0.3$, $n^* = O(H) = 3.2$. The fuzzy vertex cut (FVC) sets of Fig.4.2 are denoted such that $J_1 = \{v_6\}$ is the only 1-FVC with $s(J_1) = 0.3$. The only 2-FVC in G is $J_2 = \{v_2, v_6\}$ and $s(J_2) = 0.6$. Thus, $\kappa = 0.3$.

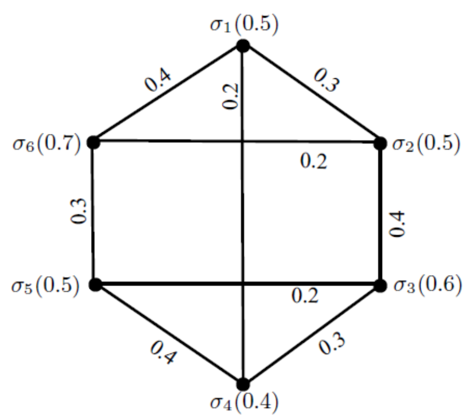

Fig.4.2 Regular fuzzy graph H with $\kappa = \vartheta_{n-1} = 0.3$

Table 4.2 : Regular fuzzy graph H to show the bounds on $\vartheta_{n-1}(H)$ and $\frac{\vartheta_1}{\vartheta_{n-1}}$

F-graph	$\vartheta_{n-1}(H)$	$\xi_1(\Delta_s, \delta_s, q)$	$\xi_2(\Delta_s, \delta_s, q)$	$\kappa(H)$	$\frac{\vartheta_1}{\vartheta_{n-1}}$	$r + \sqrt{r^2 - 1}$
H	0.3298	0.0981	0.0745	0.3	4.647	5.104

Table 4.2 shows the lower bounds on $\vartheta_{n-1}(H)$ of Theorems 4.5, 4.6 and the upper bound on $\frac{\vartheta_1}{\vartheta_{n-1}}$ of Theorem 4.7.

5. CONCLUSION

The concept of eigenvalues represented through adjacency matrices and Laplacian matrices of a fuzzy graph is discussed. The properties of the spectrum of fuzzy graphs and regular fuzzy graphs are analyzed. This study delves into the mathematical relationships between the fuzzy vertex connectivity of a fuzzy graph and its algebraic, adjacency, and Laplacian eigenvalues. These investigations yield conditions under which the fuzzy vertex connectivity is assured to be less than or equal to q. These findings make valuable contributions to the domain of spectral fuzzy graph theory, enhancing its practical applications in identifying side chain clusters within 3D protein structures, which will be discussed in forthcoming papers. This research lays the foundational knowledge for fuzzy graph theory, emphasizing spectral methods as a means to address the challenges in networks.

CONFLICT OF INTERESTS

None.

ACKNOWLEDGMENTS

The authors would like to thank the reviewers for their valuable suggestions in improving the quality of this paper.

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