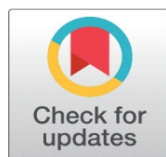
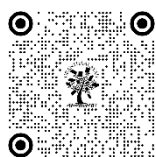


A STUDY ON SECOND GRADE MHD FLOW AND HEAT TRANSFER WITH VISCOUS AND OHMIC DISSIPATIONS OVER AN EXPONENTIALLY STRETCHING SHEET WITH

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ABSTRACT

Momentum boundary layer of viscoelastic second grade fluid taken into consideration with the effect of uniform transverse magnetic field and normal electric field also included in the analysis. Thermal boundary layer formed over the non-isothermal boundary wall, with viscous dissipation and Ohmic dissipation due to transverse magnetic field and electric field has been investigated. The highly non-linear momentum boundary layer equation and thermal boundary layer equation are converted into similarity equations and then solved numerically by employing fifth order Runge-Kutta-Fehlberg method with shooting.

Keywords: Boundary Layer, Hartman Number, Exponential Stretching and Viscous Dissipation

1. INTRODUCTION

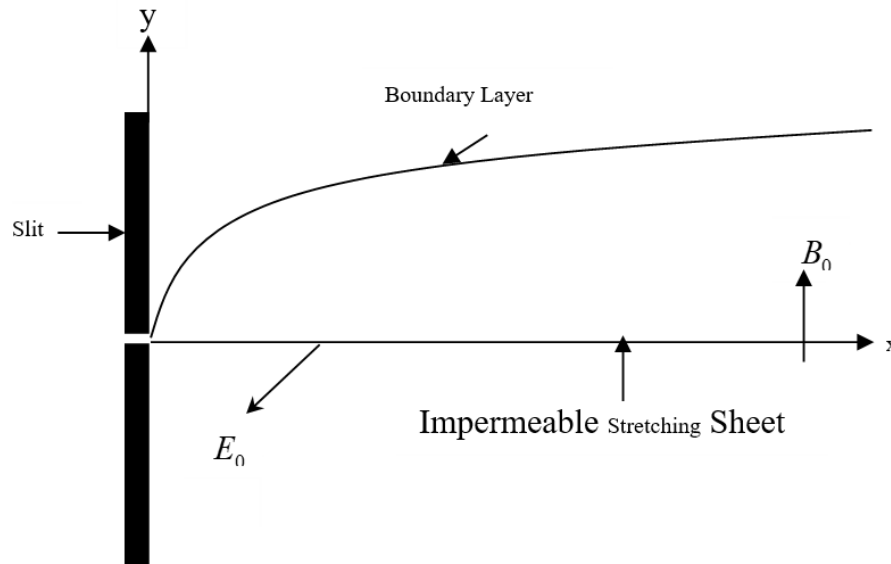
After pioneering works of Sakiadis [1964] the study over last few decades on the boundary layer flow problem over a stretching sheet as reveal many industrial applications such as polymer sheet or filament extrusion from a dye and paper production liquid films in condensation process etc. Applications are more viscoelastic in nature than viscous the studies in the subsequent years gained momentum on various aspects of momentum and heat transfer characteristics in a viscoelastic boundary layer second order fluid flow over stretching sheets Mahapatra and Gupta [2004], Pop and Soundalgekar al. [1997]. However, these studies are concerned with flow characteristics only associated with the viscoelastic fluid flow over linearly stretching sheet. In view of this some researchers have presented works on MHD flow and heat transfer in a electrically conducting viscoelastic boundary layer flow over a linear stretching sheet (Dandapat and Gupta [1989], Rollins and Vajravelu [1991]). However, in all these analysis they neglected the important controlling force, the electric field. Ali (1995) investigated thermal boundary layer by considering a power law stretching

surface. A new dimension is added to this investigation by Elbashbeshy (2001) who examined the flow and heat transfer characteristics by considering exponentially stretching sheet. However, all the above mentioned existing works on non-linear stretching sheet are confined to the viscous fluid flow in absence of magnetic field. Hence, keeping all the above mentioned works in mind present we formulate problem as follows

2. FORMULATION OF THE PROBLEM

Governing Equations

The present study, the flow is to be generated by stretching of an elastic boundary sheet with the application of two equal and opposite forces as shown in figure



We take into account the frictional heating due to viscous dissipation as the fluid considered for analysis is of viscoelastic type which possesses viscous property too. The flow region is exposed under uniform transverse magnetic fields $\vec{B}$ and uniform normal electric field $\vec{E}$. Such application of electric field and magnetic field stabilizes the boundary layer flow (Dandapat and Mukhopadhyay [2003]). The electric field $\vec{E}$ and magnetic field $\vec{B}_0$ satisfy the following MHD equations:

Maxwell's equations

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{E} = 0, \quad (2.1)$$

When magnetic field is not so strong then electric field and magnetic field obey Ohm's law. The governing boundary layer equations for momentum and heat transfers are the modified version of the equations of Andersson [1992], Khan and Sanjayanand [2005] and Cortell [1994] and Temitope E. Akinbobola, Samuel S. Okoya1(2015) and those are

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = 0 \quad (2.3)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \gamma \frac{\partial^2 u}{\partial y^2} + k_0 \left\{ u \frac{\partial^3 u}{\partial x \partial y^2} + v \frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y^2} \right\} - \frac{\sigma}{\rho} (E_0 B_0 + B_0^2 u) \quad (2.4)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{(u B_0 + E_0)^2 \sigma}{\rho c_p} \quad (2.5)$$

3. BOUNDARY CONDITIONS ON VELOCITY

$$u = U_w(x) = U_0 \exp\left(\frac{x}{l}\right), \quad v = 0 \quad \text{at} \quad y = 0 \quad (3.1)$$

$$u = 0 \quad \text{as} \quad y \rightarrow \infty.$$

Solution of the momentum boundary layer equation

To solve (2.4), we make use of the following stream function,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (3.2)$$

Stream function $\psi(x, y)$ is defined as follows

$$\psi(x, y) = \sqrt{2\gamma l U_0} f(\eta) \exp\left(\frac{x}{2l}\right)$$

$$\eta = y \sqrt{\frac{U_0}{2\gamma l}} \exp\left(\frac{x}{2l}\right)$$

Making use of the equation (3.3) in the equation (2.4)

$$f_\eta^2 - f f_{\eta\eta} = f_{\eta\eta\eta} + k_1^* \left[3f_\eta f_{\eta\eta\eta} - \frac{1}{2} f f_{\eta\eta\eta\eta} - \frac{3}{2} f_\eta^2 \right] - 2Mn^2 (E_1 + f_\eta) \quad (3.4)$$

Here $k_1^* = \frac{k_0 U_w}{\gamma l}$ is the dimensionless local viscoelastic parameter, $Mn = \sqrt{\frac{\sigma}{\rho b}} B_0$ is Hartmann number, $E_1 = \frac{E_0}{B_0 U_w}$ is

the local normal electric parameter and the subscript η stands for differentiation with respect to η .

The boundary conditions on stream function of the equation (3.1) in view of the transformations take the following non-dimensional form.

$$f(0) = 0, \quad f_\eta(0) = 1, \quad f_\eta(\infty) = 0 \quad (3.5)$$

here reasonably argued that in case of boundary layer flow of viscoelastic fluid with short memory, the characteristic time scale associated with the motion is large compared with the relaxation time of the fluid. Thus terms of order k^2 , k^3 , and higher orders may be neglected and therefore we may seek the solution of the equation (3.4) in the form

$$f = f_0(\eta) + k_1^* f_1(\eta) + O(k_1^{*2}) \quad (3.6)$$

Simplifying the equation (3.4) by using the series expansion of the equation (3.6) and equating the constant terms and the coefficient of k_1^* to zero we deduce the following equations for $f_0(\eta)$ and $f_1(\eta)$.

$$2f_{0\eta}^2 - f_0 f_{0\eta\eta} = f_{0\eta\eta\eta} - 2Mn^2 (E_1 + f_{0\eta}) \quad (3.7)$$

$$f_{1\eta\eta\eta} - 2Mn^2 f_{1\eta} - 4f_{0\eta} f_{1\eta} + f_0 f_{1\eta\eta} + f_1 f_{0\eta\eta} = 3f_{0\eta} f_{0\eta\eta\eta} - \frac{1}{2} f_0 f_{0\eta\eta\eta\eta} - \frac{3}{2} f_{0\eta}^2 \quad (3.8)$$

Substituting the series expansion of the equation (3.6) in the boundary conditions equation (3.5) we obtain boundary conditions for and in the following form.

$$f_0(0) = 0, \quad f_{0\eta}(0) = 1, \quad f_{0\eta}(\infty) = 0 \quad (3.9)$$

$$f_1(0) = 0, \quad f_{1\eta}(0) = 0, \quad f_{1\eta}(\infty) = 0 \quad (3.10)$$

The dimensionless local skin-friction coefficient is expressed as

$$C_f = - \frac{\left(\gamma \frac{\partial u}{\partial y} + k_0 \left\{ u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} - 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right\} \right)}{(bx)^2} \text{ at } y = 0 \quad (3.11)$$

$$= - \frac{1}{\sqrt{\text{Re}_x}} f_{\eta\eta}^{(0)} \left(1 - \frac{7}{2} k_1^* \right)$$

where $\text{Re}_x = \frac{U_w l}{\gamma}$ is the local Reynolds number and $f_{\eta\eta}^{(0)} = f_{0\eta\eta}^{(0)} + k_1^* f_{1\eta\eta}^{(0)}$.

4. SIMILARITY SOLUTION OF THE HEAT TRANSFER EQUATION

Case A: prescribed surface temperature (PST)

In order to solve the thermal boundary layer equation (2.5) in PST case we consider non-isothermal temperature boundary condition as follows.

$$T = T_w = T_\infty + T_0 \text{Exp} \left(\frac{\nu_0 x}{2l} \right) \quad \text{at } y = 0 \quad (4.1)$$

$$T \rightarrow T_\infty \quad \text{as } y \rightarrow \infty$$

where T_0 is the parameter of temperature distribution on the stretching surface, T_w stands for stretching sheet temperature and T_∞ is the temperature far away from the stretching sheet.

The thermal boundary layer equation (2.7) we define dimensionless temperature variable $\theta(\eta)$ of the form:

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \quad (4.2)$$

Making use of the equation (4.2) in the dimensional energy equation (2.5) we obtain thermal boundary layer equation.

$$\theta_{\eta\eta} + \text{Pr} (f \theta_\eta - \nu_0 f_\eta \theta) = - \text{Pr} E \{ f_{\eta\eta}^2 + 2 M n^2 (f_\eta^2 + E_1^2 + 2 E_1 f_\eta) \} \quad (4.3)$$

where $\text{Pr} = \frac{\gamma}{\alpha}$ is the Prandtl number and $E = \frac{U_0^2}{c_p T_0} \left(\frac{U_w}{U_0} \right)^{\frac{4-\nu_0}{2}}$ is the Eckert number.

Temperature boundary conditions of the equation (4.1) take the following non-dimensional form.

$$\theta(0) = 1, \theta(\infty) = 0 \quad (4.4)$$

The solution of the equation (4.3) subject to the boundary conditions of the equation (4.4) is obtained numerically by applying the method of fifth order Runge-Kutta-Fehlberg method with shooting and outline of the numerical solution procedure is described in the next section.

Case B: prescribed boundary heat flux (PHF)

In order to solve the thermal boundary layer equation (2.5) in PHF case we consider variable heat flux boundary condition of the following form:

$$-k \left(\frac{\partial T}{\partial y} \right)_w = T_1 \exp \left(\frac{\nu_1 + 1}{2l} \right) x \text{ at } y = 0 \quad (4.5)$$

$$T \rightarrow T_\infty \text{ as } y \rightarrow \infty$$

where is the parameter of temperature distribution on the stretching surface and T_∞ is the temperature far away from the stretching sheet. We have considered the above form of quadratic power law heat flux boundary condition on stretching sheet in order to obtain local similar solution of the equation (2.5).

As we look for local similar equation from the thermal boundary layer equation (2.7) we define dimensionless temperature variable of the form:

$$g(\eta) = \frac{T - T_\infty}{\frac{T_1}{k} \sqrt{\frac{2\gamma l}{U_0}} \exp \left(\frac{\nu_1 x}{2l} \right)} \quad (4.6)$$

Making use of the equation (4.6) in the dimensional energy equation (2.5) we arrive at the following form of non-dimensional thermal boundary layer equation in PHF case.

$$g_{\eta\eta} + Pr(fg_\eta - \nu_1 f_\eta g) = -PrE \{f_{\eta\eta}^2 + 2Mn^2(f_\eta^2 + E_1^2 + 2E_1 f_\eta)\} \quad (4.7)$$

where $Pr = \frac{\gamma}{\alpha}$ is the Prandtl number and $E = \frac{U_0^2 k}{c_p T_1 \sqrt{\frac{2\gamma l}{U_0}}} \left(\frac{U_w}{U_0} \right)^{\frac{4-\nu_1}{2}}$ is the Eckert number in PHF case.

In view of the transformation equation (4.6) the temperature boundary conditions of the equation (4.5) take the following non-dimensional form.

$$g_\eta(0) = -1, g(\infty) = 0. \quad (4.8)$$

The solution of the equation (4.7) subject to the boundary conditions of the equation (4.8) is obtained numerically by applying the fifth order Runge-Kutta-Fehlberg integration scheme with shooting and outline of the numerical solution procedure is described in the next section.

5. RESULTS AND DISCUSSION

Numerical results are computed and they are presented in the Figs. 2-15 for the case of small viscoelastic parameter in order to gain information on the effects of all the physical parameters on heat transfer characteristics. Figs 2-3 represent the stream function and velocity profiles, Figs. 4-5 represent skin-friction profiles, Figs 6-9 represent temperature profiles in PST case and Figs 10-13 represent temperature profile in PHF case for various values of heat transfer controlling parameters. The physical layout of the boundary layer in presence of electric field which develops near the slit is depicted in the

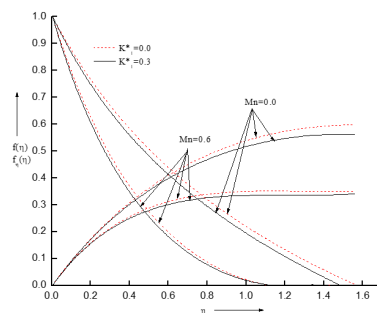


Fig.2. Graph of stream function $f(\eta)$ and velocity function $f'(\eta)$ vs. η for different values of viscoelastic parameter k^* and Hartmann number Mn when $E = 0.4$

Fig. 2. Analysis of the figure shows that the effect of increasing values of viscoelastic parameter and Hartmann number is to shift the position of streamlines towards the stretching boundary which signifies that there will be a reduction of thickness of the momentum boundary layer. The effect of viscoelastic parameter and Hartmann number is seen to decrease the boundary layer velocity throughout the boundary layer but more significantly little away from the stretching sheet. It is to be noted that this reduction of velocity with the increase of viscoelastic parameter is more significant in absence of magnetic field.

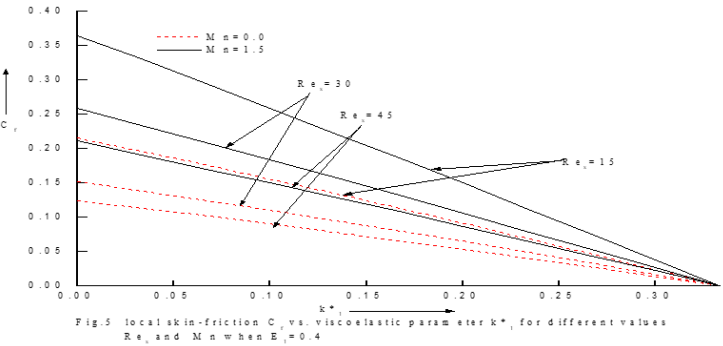


Fig. 5 represents the graphs of skin-friction coefficient C_f vs. local viscoelastic parameter k_1^* , for different values of local Reynolds number Re_x and local normal electric parameter E_1 in presence of uniform transverse magnetic field. A careful analysis of the graphs shows that the skin-friction coefficient C_f would decrease with the increase of the values of local Reynolds number Re_x . However the increase of the values of local normal electric parameter E_1 would increase the values of skin-friction parameter C_f . This is due to the fact that the Lorentz force generated by electric field coupled with the reduced magnitude of viscous force decelerates the flow in the down stream direction. It is interesting to observe that skin-friction coefficient C_f decreases linearly with the increase of viscoelastic parameter k_1^* . Separation of boundary layer occurs for the value of viscoelastic parameter $k_1^* = \frac{1}{3}$ which is independent of the values of local Reynolds number Re_x , Hartmann number Mn and local normal electric parameter E_1 . This clearly demonstrates that the effect of inertia force and electromagnetic force will be present only for such flow of fluid having smaller viscoelasticity.

Table 1

η	k_1^*	Pr	Mn	E	E_1	$\theta(\eta)$
0	0.0	2	0.0	0.0	0.0	0.99
	0.3					0.99
	0.0	5				0.981
	0.3					0.981
	0.0	5	1			0.988
	0.3					0.988
	0.0	2				0.991
	0.3					0.991
0.5	0.0	2	0.0			0.3
	0.3					0.306
	0.0	5				0.192
	0.3					0.198
	0.0	5	1			0.224
	0.3					0.237
	0.0	2				0.337
	0.3					0.35
1	0.0	2	0.0			0.3
	0.3					0.306
	0.0	5				0.192
	0.3					0.198
	0.0	5	1			0.224
	0.3					0.237
	0.0	2				0.337
	0.3					0.35

Table 2

η	k_1^*	Pr	Mn	E	E_1	$\theta(\eta)$
0	0.0	2	0.0	1	0.0	0.995
	0.3					0.996
	0.0	5				0.994
	0.3					0.995
	0.0	5	1			1.006
	0.3					1.007
	0.0	2				1.003
	0.3					1.004
0.5	0.0	2	0.0		0.511	
	0.3				0.519	
	0.0	5			0.458	
	0.3				0.466	
	0.0	5	1		0.714	
	0.3				0.716	
	0.0	2			0.72	
	0.3				0.725	
1	0.0	2	0.0		0.3	
	0.3				0.306	
	0.0	5			0.192	
	0.3				0.198	
	0.0	5	1		0.224	
	0.3				0.237	
	0.0	2			0.337	
	0.3				0.35	

The effects of various values of viscoelastic parameter k_1^* , Hartmann number Mn , Prandtl number Pr , local normal electric parameter E_1 and Eckert number E on temperature profile are tabulated in table 1 to 3 for PST case. Table 1 is tabulated for different combination of the values of viscoelastic parameter k_1^* , Hartmann number Mn and Prandtl number Pr . In absence of viscous dissipation and local normal electric field, the effect of Prandtl number Pr is to decrease temperature in the boundary layer region. Whereas, the effect of increasing values of Hartmann number Mn is to increase temperature in the boundary layer region. The boundary layer fluid would attain maximum temperature if the fluid flow is viscoelastic with low Prandtl number and it is exposed under the influence of a transverse magnetic

Table 2 tabulated for when viscous dissipation is accounted but electric field is absent ($E = 0$). In presence of viscous dissipation there would be higher temperature throughout the boundary layer in case of a viscoelastic fluid compared to viscous fluid. It is also noticed that the effect of Prandtl number is to decrease temperature throughout the boundary layer in both the cases of viscous and viscoelastic fluid in absence of magnetic field. However, in presence of magnetic field, the effect of increasing values of Prandtl number is to increase temperature near the stretching sheet and to decrease the same away from the same. The combined effect of increasing values of Prandtl number Pr and Hartmann number Mn is to increase temperature near the stretching sheet significantly.

Table 3

Table 4

η	k_1^*	Pr	Mn	E	E_1	$\theta(\eta)$
0	0.0	2	0.0	1	0.3	0.995
	0.3					0.996
	0.0	5				0.994
	0.3					0.995
	0.0	5	1			1.013
	0.3					1.014
	0.0	2				1.007
	0.3					1.008
0.5	0.0	2	0.0			0.511
	0.3					0.517
	0.0	5				0.458
	0.3					0.463
	0.0	5	1			0.992
	0.3					0.998
	0.0	2				0.93
	0.3					0.935
1	0.0	2	0.0			0.214
	0.3					0.214
	0.0	5				0.178
	0.3					0.176
	0.0	5	1			0.385
	0.3					0.397
	0.0	2				0.422
	0.3					0.426

η	k_1^*	Pr	Mn	E	E_1	$\theta(\eta)$
0	0.0	2	1.0	1	0.0	1.003
	0.3					1.004
	0.0	5			0.0	1.006
	0.3					1.076
	0.0	5			0.4	2.021
	0.3					1023
	0.0					
	0.3					
0.5	0.0	2		0.0	0.72	
	0.3			0.725		
	0.0	5	0.0	0.714		
	0.3			0.72		
	0.0	5	0.4	1.411		
	0.3			1.411		
	0.0					
	0.3					
1	0.0	2		0.0	0.288	
	0.3			0.287		
	0.0	5	0.0	0.24		
	0.3			0.24		
	0.0	5	0.4	0.695		
	0.3			0.691		
	0.0					
	0.3					

Table 3 represents the variation of temperature with the change of Hartmann number Mn , Prandtl number Pr for different values of viscoelastic parameter k_1^* when both viscous dissipation and electric field are taken into consideration. This figure is plotted for the same set of data as that of table 2 except for local normal electric field parameter $E_1=0.3$. This reveals the similar feature of temperature profile with the change of Hartmann number, Prandtl number and local viscoelastic parameter. But the region, for which temperature increases, near stretching sheet for increasing Prandtl number in presence of magnetic field, enhances. The combined effect of increasing values of Prandtl number Pr and Hartmann number Mn is to increase temperature near stretching sheet largely in presence of local normal electric field. This is owing to the reason that local normal electric field acts as a heat source in presence of magnetic field near the boundary sheet whose strength increases with the reduction of values of Prandtl number and decreases with increase of distance from the sheet.

Table 3 shows for different values of local normal electric parameter E_1 , Prandtl number Pr and viscoelastic parameter k_1^* when the effects of viscous dissipation and magnetic field are taken into account. A careful analysis table reveals that heat transfer will take place from the adjacent fluid layer to the boundary wall with lower rate when the electric field is not accounted. When the local normal electric field is accounted then the heat transfer process will

take place with faster rate. Temperature would attain maximum value in the boundary layer region near stretching sheet. This is because Ohmic dissipation due to electric as well as magnetic field is generated heat in the fluid layer near stretching sheet.

in PHF case. The analysis is carried out found the similar qualitative effects of Hartmann number Mn , Prandtl number Pr , viscoelastic parameter k_1^* , Eckert number E and local normal electric parameter E_1 on temperature profile, but with different magnitudes.

6. CONCLUSION

- 1) In presence of viscous dissipation, it is seen that there would be higher temperature throughout the boundary layer in case of a viscoelastic fluid compared to viscous fluid. The effect of Prandtl number Pr is to decrease temperature throughout the boundary layer in absence of magnetic field. However, in presence of magnetic field, the effect of increasing values of Prandtl number is to increase temperature near the stretching sheet and to decrease the same away from the same.
- 2) The combined effect of increasing values of Prandtl number Pr and Hartmann number Mn is to increase temperature near stretching sheet largely in presence of local normal electric field
- 3) When the local normal electric field is accounted then the heat transfer takes place from the adjacent fluid layer to the boundary wall with faster rate.
- 4) In PHF cases, heat transfer from stretching sheet to the adjacent boundary layer region takes place for any values of all the heat controlling physical parameters.
- 5) Magnetic field and viscous dissipation can reverse the direction of heat transfer across the stretching sheet.
- 6) The effect of viscoelastic parameter k_1^* , in presence of local electric parameter E_1 , is to increase the rate of heat transfer across the stretching sheet.

CONFLICT OF INTERESTS

None.

ACKNOWLEDGMENTS

None.

REFERENCES

- Mahaparta T. R.; Gupta. A.S.; Heat transfer in stagnation-point flow towards a stretching sheet. *Heat and Mass Transfer*. 38 (2002), 517-523.
- Pop. I.; Soundalgekar. V. M.; On thermal boundary layer of a viscoelastic fluid. *ZAMM* 57 (1997), 193-194.
- Dandapat. B. S.; Gupta A. S.; Flow and heat transfer in a viscoelastic fluid over a stretching sheet. *Int. J. Non-linear Mech.* 24, 3 (1989), 215-219.
- Rollins. D.; Vajravelu. K.; Heat transfer in a second-order fluid over a continuous stretching surface. *Acta. Mech.* 89 (1991), 167-178
- Ali ME (1995) On thermal boundary layer on a power law stretched surface with suction or injection. *Int J Heat Mass Flow* 16: 280-290
- Elbashbeshy EMA (2001) Heat transfers over an exponentially stretching continuous surface with suction. *Arch Mech* 53(6): 643-651
- Dandapat B. S.; Mukhopadhyay A.; Finite amplitude long wave instability of a film of conducting fluid flowing down an inclined plane in presence of electromagnetic field. *Int. J. Appl. Mech. Engineering*. 8 (2003) 379-393.
- Khan S. K.; Sanjayanand. E.; Viscoelastic boundary layer flow and heat transfer over an exponential stretching sheet. *Int. J. Heat and Mass Transfer* 48 (2005), 1534-1542.

- Cortell. R.; Similarity solutions for flow and heat transfer of a viscoelastic fluid over a stretching sheet; *Int. J. Non-Linear Mechanics*. 29, 2 (1994), 155-161.
- Temitope E. Akinbobola* , Samuel S. Okoya¹ The flow of second grade fluid over a stretching sheet with variable thermal conductivity and viscosity in the presence of heat source/sink *journal of Nigerian mathematic society* 34(2015) 331-342