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PAPER ON NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT

Ordinary Differential Equations (ODEs) are fundamental in modeling a wide range of phenomena in science, engineering, economics, and various other fields. However, finding exact analytical solutions to ODEs is not always possible, especially for complex, non-linear, or high-dimensional systems. In such cases, numerical methods offer a practical approach to approximating solutions. This paper explores various numerical techniques for solving ODEs, including the Euler method, Runge-Kutta methods, and multistep methods. It highlights the advantages and limitations of each approach in terms of accuracy, stability, and computational efficiency. The paper also discusses the concept of step size selection, error analysis, and the impact of discretization on the convergence of solutions. Additionally, we investigate specialized methods for stiff ODEs, which pose unique challenges due to the presence of rapidly changing variables alongside slow dynamics. Methods like the backward Euler and implicit Runge-Kutta schemes are reviewed for their effectiveness in handling stiffness. Finally, the paper presents several examples and applications to demonstrate the practical implementation of these methods and their ability to approximate solutions for real-world problems. We conclude by emphasizing the importance of choosing appropriate numerical techniques based on the nature of the problem at hand and the desired accuracy of the solution.

Keywords: Ordinary Differential Equations, Numerical Methods, Galerkin's Method, Runge-Kutta Methods, Stiff ODEs, Error Analysis, Stability

1. INTRODUCTION

Many problems in science and engineering can be reduced to the problem of solving differential equations satisfying certain given conditions. Many equations which govern physical systems does not possess closed form solutions and hence recourse must be made to be numerical methods for solving such differential equations.

There are many methods for solving ordinary differential equation, such as Taylor‟s Series, Picards Method, Euler‟s Method, Range-Kutta Method etc. Also we discuss numerical solution for boundary value problems by using finite difference method, cubic spline Method, Galarkin Method, shooting method and collocation Method etc.

Partial differential equations occur in many branches of applied Mathematics for example in hydrodynamics, elasticity, quantum mechanics and electromagnetic theory etc. The analytic treatment of these equations is rather involved process and requires application of advanced mathematical methods. On the other hand, it is generally easier to produce approximate solutions by simple and efficient numerical methods. Several numerical methods have been proposed for the solution of partial difference equations; for example finite difference method, spline method, finite element method, integral equation methods, etc. Of these, only the finite difference method have become popular and are more gainfully employed than others. In this chapter, we discuss these methods, briefly, and apply them to solve simple problems.

2. DIFFERENTIAL EQUATION

An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a differential equation.

Ex : 1) dy/dx-2xy=x^2-x

2) dy/dx+y=e^2x+x^2+1

Ordinary Differential Equation:

An equation involving ordinary derivatives of one or more dependent variables with respect to only one independent variable is called ordinary differential equations.

Ex : 1) y^''-xy^'-y=0

2) y^''-ay=0

Partial Differential Equation:

An equation involving partial derivatives of one or more dependent variable with respect to more than one independent variable is called partial differential equations.

Ex : $(\partial^2 u)/(\partial t^2)=c^2\ (\partial^2 u)/(\partial x^2)$

Initial Condition: If we have conditions put on the dependent and it's derivatives at only one value of the independent variable then they are said to be initial conditions.

Initial Value Problem: Differential equations together with an initial conditions is said to be initial value problem.

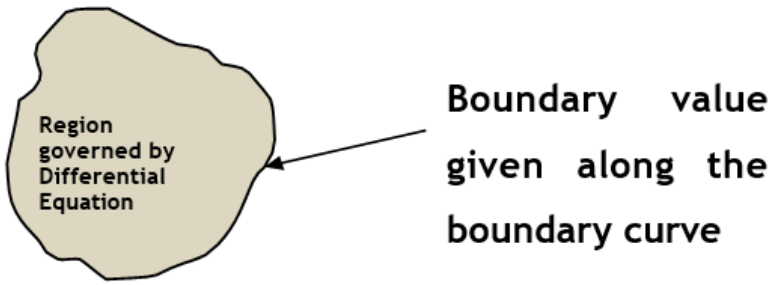
Ex: y^''+xy=e^x initial condition y(0)=1

Boundary Condition: If we have the condition on the dependent variable and its derivative at more than one point then they are called boundary condition.

The solution of Boundary value problem is a solution to a differential equation which also satisfies boundary conditions.

Ex :y^''+a_1 y^'+a_2=b with BC $y(0)=\alpha$ and $y(x)=\beta$

Boundary Value Problem: The differential equation together with boundary condition is said to be boundary value problem shows a region where a differential equations is valid with associated boundary value.



It shows a region where a differential equation is valid with associated boundary value.

Types of Boundary Condition:

Consider the two point boundary value problem

u^''=f(x,u,u^'),x∈(a,b)

Where number of primes denotes the number of times ‘ u’ is differentiable with respect to “x‟ with one of the following three conditions.

The boundary conditions are three types they are

1) Boundary condition of 1stkind (Dirichlet conditions (1)

$$u(a) = \gamma_1, \qquad u(a) = \gamma_1$$

2) Boundary Condition of 2nd kind (Neumann conditions)

$$u'(a) = \gamma_1, \qquad u'(a) = \gamma_2 \tag{2}$$

3) Boundary Condition of 3rdkind(Mixed conditions)

$$a_0 u(a) - a_1 u'(a) = \gamma_1$$
$$b_0 u(b) - b u'(b) = \gamma_2 \tag{3}$$

wherea_0, b_0,a_1, b_1,γ_1, and $\gamma_2$areconstants such that (4)

$a_0 a_1 \geq 0 \quad , \quad |a_0| + |a_1| \neq 0$

$b_0 b_1 \geq 0 \quad , \quad |b_0| + |b_1| \neq 0$

From Eq. (1) we have if all the non-zero terms involve only the dependent variable u and u' then the differential equation is called a homogenous, otherwise, it is in homogenous.

Well Posed and Ill Posed: The PDE is said to be well posed if

1) The solution is unique, when it exist.
2) The solution depends continuously on initial data.
3) Thesolutionsalwaysexistforinitialdatathatarbitrarilyclosetoinitialdataforwhichno solution exist.

Otherwise it is said to be ill posed.

Types of errors: In a numerical computation ,error may arises.

1) Truncation error
2) Round-off error

Truncation Error: Truncation error refers to the error in a method, which occurs because some series (finite or infinite) is truncated to a fewer number of terms. Such errors are essentially algorithmic errors and we can predict the extent of the error that will occur in the method.

Round-off Error: Round-off error occurs because of the computing device's inability to deal with certain numbers. Such numbers need to be rounded off to some near approximation which is dependent on the word size used to represent numbers of the device.

3. FINITE DIFFERENCE METHOD

In this method we describe the numerical solutions of two point boundary value problems by using finite difference methods. The finite difference technique is based upon approximations that allow replacing the differential equations by finite difference equation .These finite difference

Approximations are in algebraic form and the unknown solutions are related to grid points .Thus the finite difference equation basically involves three steps.

1) We define the sequence of the meshes on solution domain [a, b]
2) We approximate the given differential equation by the system of difference equations that relates the solutions to grid points
3) We solve the above algebraic system of the equations.

Finite difference method for the solution of a two point boundary problem consists on replacing the derivatives occurring in the differential equation by means of their finite difference approximations and then solving the resulting linear system of equations by a standard procedure.

$$y'' + p(x)y' + q(x)y = r(x)$$

$$y(x_0) = a \; and \; y(x_n) = b$$

To obtain the approximations to the derivatives we proceed as follows by expanding y(x+h) in Taylor's series we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \ldots\ldots\ldots\ldots\ldots \quad (1)$$

From which we obtain ,

$$y'(x) = \frac{y(x+h)+y(x)}{h} - \frac{h^2}{2}y''(x) \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Which is a forward difference approximation, similarly expanding y(x-h) in taylor's series, we have

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$y'(x) = \frac{y(x)-y(x-h)}{h} - 0(h) \quad (4)$$

A central difference approximation is obtained by subtracting Eq.(1)-Eq.(3) we have

$$y'(x) = \frac{y(x+h)+y(x-h)}{2h} - 0(h^2) \quad (5)$$

Again add Eq.(1) and Eq.(2) we get approximation for

$$y''(x) = \frac{y(x+h)-2y(x)+y(x-h)}{h^2} - 0(h^2) \qquad (6)$$

similarly, it is possible to derive finite difference approximation to higher derivative. To solve the boundary value problem we divide the range [X0, Xn] in to 'n' subintervals of width 'h' so that

$x_i = x_0 + ih \qquad y(x_i) = y(x_0 + ih), \quad where\ i = 0,1,2 \ldots\ldots n$

From Eq.(5) and Eq.(6) , we have

$y'_i = \frac{y_{i+1}+y_{i-1}}{2h} + 0(h^2) \qquad and \qquad y_i = \frac{y_{i-1}-2y_i+y_{i+1}}{h^2} + 0(h^2)$

Satisfying differential at point $x = x_i$_i, we get

$y_i + f_i g_i + g_i y_i = r_i$

Substituting the expression for $y'\ and\ y''$t this gives

$\frac{y_{i-1}-2y_i+y_{i+1}}{h^2} + f_i\left(\frac{y_{i+1}-y_{i-1}}{2h}\right) + g_i y_i = r_i y^n$

Where $y_i = y(x_i),\ \ g_i = g(x_i), i = 1,2,3, \ldots . n$ multiplying through by h^2 and simplifying , we obtain

$\left(1 - \frac{h}{2} f_i\right) y_{i-1} + (-2 + g_i h^2) y_i + \left(1 + \frac{h}{2} f_i\right) y_{i+1} = r_1 h^2$

Where $i = 1,2,3, \ldots . n-1\ with\ y_0 = a\ and\ y_n$

1) Galerkin'sMethod:

Galerkin's Method is also called the weighted residual method, (trial function (or approximating functions) which satisfy the boundary conditions of the problem). The trial function is substituted in the given differential equation and the result is called the residual. The integral of the product of this residual and a weighted function, taken over the domain, is then set to zero which yields a system of equations for the unknown parameters in the trial functions.

Let the boundary value problem be defined by

$$y'' + p(x)y' + q(x)_y = f(x) \qquad a < x < b \qquad (1)$$

With boundary condition

$$p_0 y(a) + q_0 y'(a) = r_0 \qquad p_1 y(b) + q_1 y'(b) = r_1 \qquad (2)$$

Let the approximate solution be given by

$$t(x) = \sum_{i=1}^{n} \alpha_i \emptyset_i(x) \qquad (3)$$

Where ∅_i (x) are called base functions . Substituting for t(x) in equation (1) we obtain a residual denoting this residual by R(t) we obtain

$$R(t) = t'' + p(x)t' + q(x)t - f(x) \qquad (4)$$

Usually the functions ∅_i (x) are chosen as weight functions we have

$$I = \int_a^b \emptyset_i(x) R(t) dx = 0 \qquad (5)$$

Which yields a system of equations for the parameters α_i. When α_i are known t(x) can be calculated from equation (3)

Weighted Residual Method:

The method of weighted residuals (MWR) is an approximate technique for solving boundary value problems that utilize trial functions satisfying the prescribed boundary conditions an integral formulation to minimize error. The general concepts is described here in terms of the one-dimensional case but as shown in later extension for two and three dimensions. Which gives a differential equation of the general form.

$D[y(x), x] = 0\,, \qquad 0 \le x \le 1$ (1)

Subjected to (homogeneous) boundary conditions

$y(0) = y(1) = 0$ (2)

The method of weighted residuals seeks an approximate solution in the form

$y^*(x) = \sum_{i=1}^{n} c_i N_i(x)$ (3)

Where y^* is the approximate solution expressed as the product of C_i unknown , constant parameters to be determined and N_i trail functions and weighted residuals is obtained from Eq.(3) is simply called residuals, such as

$R(x) = D[y^*(x), x] \neq 0$ (4)

Where R(x) is the residual note that the residual is also a function of the unknown parameters C_i be evaluated such that

$\int w_i(x) R(x) dx = 0\ , \quad i = 1,2,3 \dots \dots \dots n$ (5)

Where w(x) represents n , arbitrary weighting functions we observe that on integration equations which can be solved for 'n' values of C_i , equation (5) expresses that the sum (integral) of the weighted residual error over the domain of the problem is zero.

Several variations of MWR exist and the techniques vary primarily in how the weighting factors are determined or selected. The most common techniques are point collocation, sub-domain collocation, least squares and Galerkin's method. We discuss only Galerkin's method it is quite simple to use and readily adaptable to the finite element method in galerkin's weighted residual method, the weighting functions are chosen to be identical to the trial functions that is

$w_i(x) = N_i(x)\,, \quad i = 1,2,3, \dots \dots \dots n$ (6)

Therefore the unknown parameters are determined via

$\int_a^b w_i(x) R(x) dx = \int_a^b N_i(x) R(x) dx$ (7)

Again resulting in 'n' algebraic equations for evaluation of the unknown parameters. The following examples illustrate details of procedure.

Ex.1 : Use galerkin's method of weighted residual to obtain an approximate solution of the differential equations.

$\frac{d^2y}{dx^2} - 10x^2 = 5 \quad 0 \le x \le 1$

With the boundary conditions $y(0) = y(1)$

Soln: The presence of the quadratic terms in differential equations suggest that trail functions in polynomial form are suitable for homogeneous boundary conditions at x=0 and x=b the general forms

$N(x) = (x - x_a)^p (x - x_b)^q$

With 'p' and 'q' being +ve integers greater than zero automatically satisfies the boundary conditions and is continues in $x_a \le x \le x_b$ using a single trail function . The simplest such form that satisfies the stated the boundary

conditions $N_1(x) = x(x-1)$. Using the trail function the approximate solution for equation (3) , $y^*(x) = c_1 x(x-1)$and the first and second derivatives are

$\frac{dy^*}{dx} = c_1(2x-1)$

$\frac{d^2y^*}{dx^2} = 2c_1$

Substitute of the second derivative of y^* (x) in to the differential equation yields the residual as

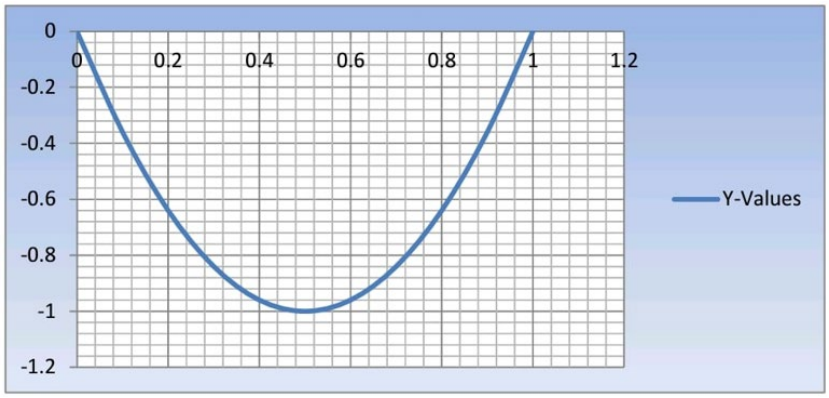
$R(x; c_1) = 2c_1 - 10x^2 - 5$

Which is clearly non zero , substitution in to equation (7)

$\int x(x-1)(2c_1 - 10x^2 - 5)dx = 0$

Which after integration yields c_1=4 so the approximate solution is obtained as

$y^*(x) = 4x(x-1)$



Ex.2: Obtain a two term galerkin solution problem of $D[y(x), x] = 0$, $a < x < b$ subjected to homogenous boundary conditions using trail functions $N_1(x) = x(x-1)$ $N_2(x) = x^2(x-1)$

Soln: The two-term approximated solution is

$y^* = c_1 x(x-1) + c_2 x^2(x-1)$

And the second derivatives is

$\frac{dy^*}{dx} = 2c_1 + 2c_2(3x-1)$

Substituting into the differential equation , we obtain the residual

$R(x, c_1, c_2) = 2c_1 + 2c_2(3x-1) - 10x^2 - 5$

Using the trail functions as the weighting function for Galerkin's method the residual equations becomes

$\int_0^1 x(x-1)[2c_1 + 2c_2(3x-1) - 10x^2 - 5]dx = 0$

$\int_0^1 x^2(x-1)[2c_1 + 2c_2(3x-1) - 10x^2 - 5]dx = 0$

After integration and simplification , we obtain the algebraic equations

$-\frac{c_1}{3} - \frac{c_2}{6} + \frac{4}{3} = 0$

$$-\frac{c_1}{6}-\frac{c_2}{15}+\frac{4}{4}=0$$

Simultaneous solution results in

$$c_1=\frac{19}{6}\ ,\quad c_2=\frac{5}{3}$$

So the two terms approximation solution is

$$y^*=\frac{19}{6}x(x-1)+\frac{5}{3}x^2(x-1)=\frac{5}{3}x^3+\frac{3}{2}x^2-\frac{19}{6}x$$

4. CONCLUSION

Numerical methods are essential for solving ODEs that do not have analytical solutions. Methods such as Euler's, Runge-Kutta, and implicit methods provide a range of tools for approximating the solution to an ODE. The choice of method depends on the specific problem at hand, such as the need for accuracy, stability, and the nature of the ODE (whether it is stiff or non-stiff). In practice, the trade-off between computational cost and accuracy is a key consideration when choosing a numerical method for solving ODEs. Numerical methods continue to be a cornerstone of applied mathematics, helping scientists and engineers solve complex real-world problems that cannot be addressed through analytical methods alone.The Galerkin method is a powerful and versatile numerical technique for solving ODEs, especially in problems with boundary or initial conditions. It provides a systematic way to convert an ODE into a solvable system of equations through a choice of appropriate basis functions.While the method offers significant advantages in terms of flexibility, systematic approximation, and convergence, its application can be computationally expensive for large systems or nonlinear ODEs. The success of the Galerkin method depends on the careful selection of basis functions and the handling of boundary conditions.

CONFLICT OF INTERESTS

None.

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None.

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